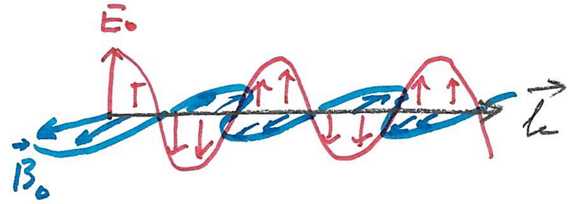
Homework Assignment 3

Problem 3-1

(A)



(B) $\vec{E} = \vec{E_0} \cos(kz - \omega t)$ and $\vec{B} = \vec{B}_0 \cos(kz - \omega t)$

where $\omega = ck$ and $B_0 = E_0$.

$$u_E = \frac{1}{8\pi} E^2 = \frac{E_0^2}{8\pi} \cos^2(kz - \omega t)$$

Check units: $(M^{1/2} L^{-1/2} T^{-1})^2 = \frac{ML^2/T^2}{L^3}$ ✓

$$u_B = \frac{1}{8\pi} B^2 = \frac{B_0^2}{8\pi} \cos^2(kz - \omega t)$$

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} = \frac{c}{4\pi} E_0 B_0 \hat{k} \cos^2(kz - \omega t)$$

Local conservation of energy:

- $\frac{\partial u}{\partial t} = \frac{E_0^2}{4\pi} 2\cos(kz - \omega t)\, \omega \sin(kz - \omega t) = \frac{E_0^2 \omega}{2\pi} \cos(\Phi)\sin(\Phi)$
- $\nabla \cdot \vec{S} = \frac{c}{4\pi} E_0^2 \; 2\cos(kz - \omega t)(-k) \sin(kz - \omega t) = -\frac{E_0^2 \omega}{2\pi} \cos(\Phi)\sin(\Phi)$

So $\frac{\partial u}{\partial t} + \nabla \cdot \vec{S} = 0$ ✓

(C) Radio, Microwaves, IR, Visible, UV, Xrays, γrays

(D) Any sum of solutions of the field equations is also a solution.

(E) Any solution of the field equations can be written as a sum of plane waves.

Problem 3-2

Given $\vec{E}(\vec{x},t) = \mathrm{Re}\left[\vec{E}_0\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}\right]$

$\vec{B}(\vec{x},t) = \mathrm{Re}\left[\vec{B}_0\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}\right]$

where $\omega = ck$ and $B_0 = E_0$.

(a) A point with constant phase has

$$kx - \omega t = \text{constant}$$

$$\therefore\ k\,\delta x - \omega\,\delta t = 0$$

Thus the phase velocity is $\dfrac{\delta x}{\delta t} = \dfrac{\omega}{k}$

(b) The dispersion relation $\omega = ck$ follows from Maxwell's equations. Thus $v_{phase} = c$.

Homework Assignment #3

Problem 3-3 (Exercise 1.4.2)

(a) Let $\vec{F} = \vec{e}\phi$ where $\vec{e}$ is an arbitrary vector.

Gauss's Law: $\int_V d^3x\, \nabla\cdot\vec{F} = \oint_S da\, \hat{n}\cdot\vec{F}$

left side $= \int_V d^3x\, \vec{e}\cdot\nabla\phi = \vec{e}\cdot\int_V d^3x\, \nabla\phi$

right side $= \oint_S da\, \hat{n}\cdot\vec{e}\phi = \vec{e}\cdot\oint_S da\, \hat{n}\phi$

Since left = right for any vector $\vec{e}$,

$$\int_V d^3x\, \nabla\phi = \oint_S da\, \hat{n}\,\phi \quad ✓$$

(b) Let $\vec{G} = \vec{e}\times\vec{A}$.

Gauss law: $\int_V d^3x\, \nabla\cdot\vec{G} = \oint_S da\, \hat{n}\cdot\vec{G}$

left side $= \int d^3x\, \nabla\cdot(\vec{e}\times\vec{A}) = \int d^3x\, \epsilon_{ijk}\frac{\partial}{\partial x_i} e_j A_k(x)$

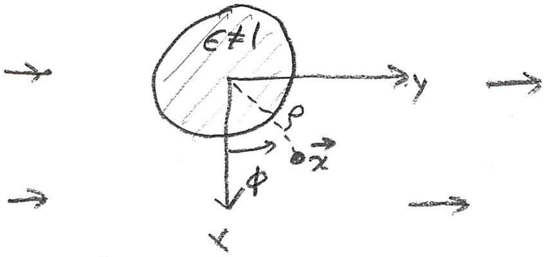
$= -\vec{e}\cdot\int d^3x\, \nabla\times\vec{A}$

right side $= \oint da\, \hat{n}\cdot(\vec{e}\times\vec{A}) = -\vec{e}\cdot\oint da\, (\hat{n}\times\vec{A})$

Since left = right for any vector $\vec{e}$,

$$\int_V d^3x\, \nabla\times\vec{A} = \oint_S da\, \hat{n}\times\vec{A}. \quad ✓$$

Problem 3-4 (Exercise 5.4.3)



$\vec{E} = E_0 \hat{e}_y$ = the initial electric field
= the asymptotic field.

Dielectric cylinder in an initially uniform field $E_0 \hat{e}_y$

The field equations are $\nabla \times \vec{E} = 0$ and $\nabla \cdot \vec{D} = 0$ where $\vec{D} = \epsilon \vec{E}$. Therefore

$\vec{E} = -\nabla \Phi$ and $\nabla^2 \Phi = 0$.

The general solution of Laplace's equation for cylindrical symmetry is (eq. 3.118)

$$\Phi(\rho, \phi) = a_0 + b_0 \ln \frac{\rho}{\rho_0} + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha_n) + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin(n\phi + \beta_n)$$

For this problem the solution must have the form

$$\Phi_{in}(\vec{x}) = a_{11} \rho \sin\phi \qquad (\rho < R)$$

$$\Phi_{out}(\vec{x}) = a_{12} \rho \sin\phi + b_1 \frac{1}{\rho} \sin\phi \qquad (\rho > R)$$

Now impose the boundary conditions at $\rho = R$:

E_ϕ is continuous $\Rightarrow$ $a_{11} R = a_{12} R + \dfrac{b_1}{R}$

D_ρ is continuous $\Rightarrow$ $\epsilon a_{11} = a_{12} - b_1/R^2$

The asymptotic potential is $-E_0 y = -E_0 r\sin\phi$,

so $a_{12} = -E_0$.

Two equations for 2 unknowns,

$$\left.\begin{aligned} a_{11} - \frac{b_1}{R^2} &= -E_0 \\ \epsilon a_{11} + \frac{b_1}{R^2} &= -E_0 \end{aligned}\right\} \text{ Solutions } \left\{\begin{aligned} a_{11} &= \frac{-2E_0}{\epsilon+1} \\ \frac{b_1}{R^2} &= \frac{\epsilon-1}{\epsilon+1} E_0 \end{aligned}\right.$$

Thus

$$\Phi_{in}(\bar{x}) = \frac{-2E_0}{\epsilon+1}\rho\sin\phi \qquad (\rho < R)$$

$$\Phi_{out}(\bar{x}) = -E_0\rho\sin\phi + \frac{\epsilon-1}{\epsilon+1}\frac{R^2}{\rho}\sin\phi \qquad (\rho > R)$$

Problem 3-5 (Exercise 9.1.1)

Given harmonic waves in medium (ϵ, μ)

$\vec{E}(\vec{x},t) = \mathrm{Re}\left(\vec{\mathcal{E}}\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}\right)$ and $\vec{B}(\vec{x},t) = \mathrm{Re}\left[\vec{\mathcal{B}}\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}\right]$

Maxwell's equations:

$$\nabla\cdot\vec{D} = 0 \Rightarrow \vec{k}\cdot\vec{\mathcal{E}} = 0 \; ; \quad \nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \Rightarrow \vec{k}\times\vec{\mathcal{E}} = \frac{\omega}{c}\vec{\mathcal{B}}$$

$$\nabla\cdot\vec{B} = 0 \Rightarrow \vec{k}\cdot\vec{\mathcal{B}} = 0 \; ; \quad \nabla\times\vec{H} = \frac{1}{c}\frac{\partial\vec{D}}{\partial t} \Rightarrow \underbrace{\frac{\vec{k}\times\vec{\mathcal{B}}}{\mu} = -\frac{\omega}{c}\epsilon\vec{\mathcal{E}}}$$

$$\vec{k}\times(\vec{k}\times\vec{\mathcal{B}}) = -k^2\vec{\mathcal{B}}$$
$$= -\frac{\mu\omega\epsilon}{c}\frac{\omega}{c}\vec{\mathcal{B}}$$
$$\therefore \quad c^2k^2 = \mu\epsilon\omega^2$$

(a) The time averaged energy densities:

$$w_e = \frac{1}{16\pi}\vec{\mathcal{E}}\cdot\mathcal{D}^* = \frac{\epsilon}{16\pi}|\vec{\mathcal{E}}|^2$$

$$w_m = \frac{1}{16\pi}\vec{\mathcal{B}}\cdot\vec{\mathcal{H}}^* = \frac{1}{16\pi\mu}|\vec{\mathcal{B}}|^2$$

$$= \frac{1}{16\pi\mu}\frac{c}{\omega}(\vec{k}\times\vec{\mathcal{E}})\cdot\frac{c}{\omega}(\vec{k}\times\vec{\mathcal{E}}^*)$$

$$= \frac{c^2}{16\pi\mu\omega^2}k^2|\vec{\mathcal{E}}|^2 = \frac{\mu\epsilon\omega^2}{16\pi\mu\omega^2}|\vec{\mathcal{E}}|^2 = \frac{\epsilon}{16\pi}|\vec{\mathcal{E}}|^2$$

$$w_m = w_e \quad \checkmark$$

(b) Calculate $\vec{S} = \frac{c}{8\pi}\vec{\mathcal{E}}\times\vec{\mathcal{H}}^*$ (harmonic energy flux)

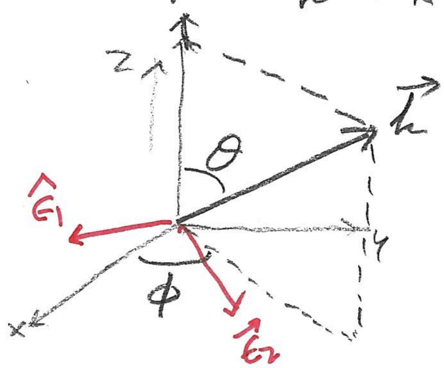
$$\vec{S} = \frac{c}{8\pi}\vec{\mathcal{E}}\times\frac{1}{\mu}\frac{c}{\omega}(\vec{k}\times\vec{\mathcal{E}}^*) = \frac{c^2}{8\pi\mu\omega}\vec{k}|\vec{\mathcal{E}}|^2$$

$$\boxed{\begin{array}{l} \bullet\; v_{phase} = \frac{\omega}{k} = \frac{c}{\sqrt{\mu\epsilon}} \\ \bullet\; |\vec{\mathcal{E}}|^2 = \frac{16\pi}{\epsilon}w_e \\ \quad = \frac{8\pi}{\epsilon}w_{em} \end{array}}$$

$$\vec{S} = \frac{c^2}{8\pi\mu\omega}\frac{8\pi}{\epsilon}w_{em}\frac{\omega\hat{k}}{v_{phase}} = v_{phase}\, w_{em}\,\hat{k}.$$

$$\boxed{c^2 = \mu\epsilon v_{phase}^2}$$

(c) Suppose $\vec{k} = k \left\{ \begin{array}{ccc} (x) & (y) & (z) \\ \sin\theta\cos\phi, & \sin\theta\sin\phi, & \cos\theta \end{array} \right\}$



• $\hat{\epsilon}_1$ is in the xy plane.

$\hat{\epsilon}_1 = \hat{e}_x \cos\alpha + \hat{e}_y \sin\alpha$

$$\begin{aligned} \hat{\epsilon}_1 \cdot \vec{k} &= k\left[\sin\theta\cos\phi\cos\alpha + \sin\theta\sin\phi\sin\alpha \right] \\ &= k\sin\theta\cos(\phi - \alpha) \\ &= 0 \Rightarrow \alpha = \phi + \pi/2 \end{aligned}$$

$$\hat{\epsilon}_1 = -\hat{e}_x \sin\phi + \hat{e}_y \cos\phi$$

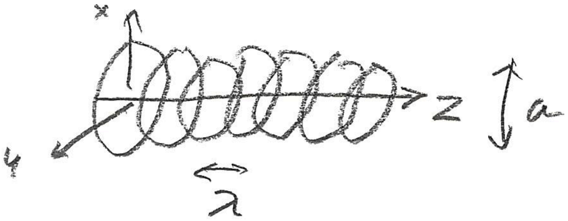
• $\hat{\epsilon}_1 \times \hat{\epsilon}_2 = \hat{k} \Rightarrow \hat{k} \times \hat{\epsilon}_1 = \hat{\epsilon}_2$

$$\therefore \hat{\epsilon}_2 = \begin{bmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ s_\theta c_\phi & s_\theta s_\phi & c_\theta \\ -s_\phi & c_\phi & 0 \end{bmatrix} = \left\{ \begin{array}{ccc} (x) & (y) & (z) \\ -c_\theta c_\phi, & c_\theta s_\phi, & s_\theta \end{array} \right\}$$

$$\hat{\epsilon}_2 = -\hat{e}_x \cos\theta\cos\phi + \hat{e}_y \cos\theta\sin\phi + \hat{e}_z \sin\theta$$

Problem 3-6 (Exercise 9.1.3)

E.M. wave with finite extent, a.



assume $\lambda \ll a$

I.e., $k = \frac{2\pi}{\lambda} \gg 2\pi/a$

(a) $\vec{E}(\vec{x},t) \approx \left[\vec{\mathcal{E}}_0(\vec{x}_T) + \frac{i}{k}\hat{e}_3 \, \nabla\cdot\vec{\mathcal{E}}_0(\vec{x}_T)\right] e^{i(kz-\omega t)}$

where $\vec{x}_T = x\hat{e}_x + y\hat{e}_y$.

We must have $\nabla\cdot\vec{E} = 0$. $\qquad \nabla_T = \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial z}$

Calculate $\quad \nabla\cdot\vec{E} = \frac{\partial}{\partial z} E_3 + \nabla_T\cdot\vec{E}_T$

$$= \left[\mathcal{E}_{0z} + \frac{i}{k}\nabla_\perp\cdot\vec{\mathcal{E}}_0\right] ik\, e^{i(kz-\omega t)}$$

$$+ \left[\nabla_T\cdot\vec{\mathcal{E}}_0\right] e^{i(kz-\omega t)}$$

$$= \mathcal{E}_{0z}\, ik\, e^{i(kz-\omega t)}$$

$= 0$ requires $\mathcal{E}_{0z} = 0$.

(b) Now calculate $\vec{B}$.

By Faraday's law, $\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} = \frac{i\omega}{c}\vec{B}$

$$\vec{B} = -i\frac{c}{\omega}\nabla\times\vec{E} = \frac{-ic}{\omega}\nabla\times\left\{\left[\vec{\mathcal{E}}_0 + \frac{i}{k}\hat{e}_3\nabla\cdot\vec{\mathcal{E}}_0\right]e^{i(kz-\omega t)}\right\}$$

$$= \frac{-ic}{\omega}\left\{\nabla\times\left[\vec{\mathcal{E}}_0 + \frac{i}{k}\hat{e}_3\nabla\cdot\vec{\mathcal{E}}_0\right]\times e^{i(kz-\omega t)} - \left[\vec{\mathcal{E}}_0 + \frac{i}{k}\hat{e}_3\nabla\cdot\vec{\mathcal{E}}_0\right]\times\nabla e^{i(kz-\omega t)}\right\}$$

$$= \frac{-ic}{\omega}\left\{\nabla\times\vec{\mathcal{E}}_0 + \nabla\times\frac{i}{k}\hat{e}_3\nabla\cdot\vec{\mathcal{E}}_0 - \left[\vec{\mathcal{E}}_0 + \frac{i}{k}\hat{e}_3\nabla\cdot\vec{\mathcal{E}}_0\right]ik\hat{e}_3\right\}e^{i(kz-\omega}$$

$$= \frac{-ic}{\omega}e^{i(kz-\omega t)}\left\{\nabla\times\vec{\mathcal{E}}_0 + \frac{i}{k}\nabla\times(\hat{e}_3\nabla_T\cdot\vec{\mathcal{E}}_0) - ik\,\vec{\mathcal{E}}_0\times\hat{e}_3\right\}$$

$$\begin{vmatrix}\hat{e}_1 & \hat{e}_2 & \hat{e}_3\\ \partial_x & \partial_y & 0\\ \mathcal{E}_{0x} & \mathcal{E}_{0y} & 0\end{vmatrix} = \hat{e}_3\left(\frac{\partial\mathcal{E}_{0y}}{\partial x} - \frac{\partial\mathcal{E}_{0x}}{\partial y}\right)$$

$\mathcal{E}_{0z} = 0$ by part (a)

$$= \frac{-ic}{\omega}e^{i(kz-\omega t)}\left\{ik\,\hat{e}_3\times\vec{\mathcal{E}}_0 + \hat{e}_3\left(\frac{\partial\mathcal{E}_{0y}}{\partial x} - \frac{\partial\mathcal{E}_{0x}}{\partial y}\right) + \frac{i}{k}\nabla_T\times(\hat{e}_3\nabla_T\cdot\vec{\mathcal{E}}_0)\right\}$$

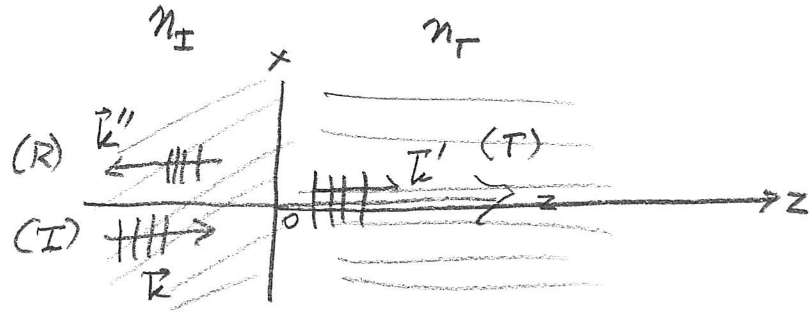
Compare: $\hat{e}_3\nabla\cdot(\hat{e}_3\times\underset{\downarrow\ \mathcal{E}_{0x}\hat{e}_x + \mathcal{E}_{0y}\hat{e}_y}{\vec{\mathcal{E}}_0}) = \hat{e}_3\nabla\cdot(\hat{e}_y\mathcal{E}_{0x} - \hat{e}_x\mathcal{E}_{0y})$

$$= \hat{e}_3\left(\frac{\partial\mathcal{E}_{0x}}{\partial y} - \frac{\partial\mathcal{E}_{0y}}{\partial x}\right)$$

$$\therefore\ \vec{B} = \frac{-ic}{\omega}e^{i(kz-\omega t)}\left\{ik\,\hat{e}_3\times\vec{\mathcal{E}}_0 - \hat{e}_3\nabla\cdot(\hat{e}_3\times\vec{\mathcal{E}}_0) + \frac{i}{k}\nabla_T\times(\hat{e}_3\nabla\cdot\vec{\mathcal{E}}_0)\right\}$$

$$= \underbrace{\frac{ck}{\omega}e^{i(kz-\omega t)}\left\{\hat{e}_3\times\vec{\mathcal{E}}_0 + \frac{i}{k}\hat{e}_3\nabla\cdot(\hat{e}_3\times\vec{\mathcal{E}}_0)\right.}_{} \left. + O(1/k^2)\right\}$$

Problem 3-7



Reflection and Refraction at Normal Incidence

There are no normal component of $\vec{E}$ or $\vec{B}$.
(The normal vector is $\hat{n} = \hat{e}_z$.) Assume (WLOG) $\vec{E} = E_0 \hat{e}_x e^{i(kz-\omega t)}$

$\vec{E}_{tangential}$ is continuous $\Rightarrow$ $E_0 + E_0'' = E_0'$

$\vec{H}_{tangential}$ is continuous $\Rightarrow$ $\frac{ck}{\omega} E_0 - \frac{ck''}{\omega} E_0'' = \frac{ck'}{\omega} E_0'$

(Note $\vec{B} = \frac{c}{\omega} \vec{k} \times \vec{E}$ by Faraday's Law)

Now, $\frac{ck}{\omega} = \frac{c}{v_{phase}} = n_I$

and $\frac{ck'}{\omega} = \frac{c}{v'_{phase}} = n_T$

$$\begin{cases} E_0' - E_0'' = E_0 \\ n_T E_0' + n_I E_0'' = n_I E_0 \end{cases} \Rightarrow \begin{cases} E_0' = \dfrac{2 n_I}{n_I + n_T} E_0 \\ E_0'' = \dfrac{n_I - n_T}{n_I + n_T} E_0 \end{cases}$$

(B) TE polarization and TM polarization are <u>the same</u> for normal incidence.

(C) and (D) See the graphs.

$\nabla \times \vec{E} = -\partial \vec{B}/\partial t \Rightarrow i\vec{k} \times \vec{E} = i\omega \vec{B} \Rightarrow \vec{B} = \frac{c\vec{k}}{\omega} \times \vec{E} = \frac{ck}{\omega} \hat{e}_y E_0$