Problem 6-1 Exercise 9.9.1

Eq. 9.269 $\quad \lim_{\omega\to\infty} \frac{1}{\epsilon_{eff}(\omega)} \to 1 + \frac{\omega_p^2}{\omega^2}$ ← defined ω_p^2

Eq 9.271 $\quad \omega_p^2 = -\frac{2}{\pi}\int_0^\infty d\omega\, \omega\, \mathrm{Im}\left(\frac{1}{\epsilon_{eff}}\right)$ ← prove this, assuming $\frac{1}{\epsilon_{eff}(\omega)}$ has no poles in the UHωP.

Proof

$$0 = \frac{1}{2\pi}\oint \frac{dz}{z-\omega+i\epsilon}\left[\frac{1}{\epsilon_{eff}(z)} - 1\right]$$ because $\frac{1}{\epsilon_{eff}(z)}$ has no poles in the UHzP.

$$0 = \int_{-\infty}^{\infty} \frac{d\omega'}{(2\pi)(\omega'-\omega+i\epsilon)}\left[\frac{1}{\epsilon_{eff}(\omega')} - 1\right]$$ because semi circle at ∞ contribution is 0

$$\Rightarrow \quad \mathrm{Re}\,\frac{1}{\epsilon_{eff}(\omega)} - 1 = \frac{2}{\pi} P\int_0^\infty d\omega' \frac{\omega'\, \mathrm{Im}\,\epsilon_{eff}^{-1}(\omega')}{\omega'^2-\omega^2}$$

$$\mathrm{Im}\,\frac{1}{\epsilon_{eff}(\omega)} = \frac{-2\omega}{\pi} P\int_0^\infty d\omega' \frac{\mathrm{Re}\,\epsilon_{eff}^{-1}(\omega') - 1}{\omega'^2-\omega^2}$$

(same derivation as for the Kramers Kronig relations)

Then $$\omega_p^2 \equiv \lim_{\omega\to\infty} \omega^2\left[\frac{1}{\epsilon_{eff}(\omega)} - 1\right]$$

$$= \frac{2}{\pi} P\int_0^\infty d\omega' \underbrace{\frac{\omega^2}{\omega'^2-\omega^2}}_{\to -1 \text{ as } \omega\to\infty} \omega'\, \mathrm{Im}\,\epsilon_{eff}^{-1}(\omega')$$

$$\omega_p^2 = -\frac{2}{\pi}\int_0^\infty d\omega'\, \omega'\, \mathrm{Im}\,\epsilon_{eff}^{-1}(\omega')$$

as claimed

Part b Assume $\epsilon(\omega) = 1 + \frac{4\pi e^2}{m} \frac{n_b}{\omega_0^2 - \omega^2 - i\omega\gamma}$

(The "one resonance expression")

We want to prove that $\omega_p^2 = \frac{-2}{\pi}\int_0^\infty d\omega\, \omega\, \mathrm{Im}\left(\frac{1}{\epsilon(\omega)}\right)$

(equation 9.271)

Calculation

$$\frac{1}{\epsilon(\omega)} = \frac{1}{1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\gamma}} \qquad \text{where } \omega_p^2 = \frac{4\pi e^2}{m} n_b$$

$$= \frac{\omega_0^2 - \omega^2 - i\omega\gamma}{\omega_p^2 + \omega_0^2 - \omega^2 - i\omega\gamma} \times \frac{\omega_p^2 + \omega_0^2 - \omega^2 + i\omega\gamma}{\omega_p^2 + \omega_0^2 - \omega^2 + i\omega\gamma}$$

$$i\,\mathrm{Im}\left(\frac{1}{\epsilon(\omega)}\right) = \frac{-i\omega\gamma[\omega_p^2 + \omega_0^2 - \omega^2] + i\omega\gamma[\omega_0^2 - \omega^2]}{(\omega_p^2 + \omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

$$\mathrm{Im}(1/\epsilon) = \frac{-\omega\gamma\,\omega_p^2}{(\omega_p^2 + \omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

Now calculate

$$-\frac{2}{\pi}\int_0^\infty d\omega\, \omega\, \mathrm{Im}\left(\frac{1}{\epsilon}\right) = +\frac{2}{\pi}\int_0^\infty d\omega \frac{\gamma\omega_p^2\,\omega^2}{(\omega_p^2 + \omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

To make life simple, I'll calculate the integral using MATHEMATICA.

Result

$$-\frac{2}{\pi}\int_0^\infty d\omega\, \omega\, \mathrm{Im}(1/\epsilon) = \omega_p^2$$

So that proves part (b).

Problem 6-2   Exercise 9.9.2

Stopping range $R(W_0)$ for an electron with initial kinetic energy $W_0 = \frac{1}{2} m v_0^2$ in aluminum.

Start with $$-\frac{dE}{dx} = \frac{4\pi N e^4}{m v^2} \ln\left(\frac{5 m v^2}{\hbar \bar{\omega}}\right)$$

(Given Parameters: $Z = 13$, $M = 27\,g/mol$, $\rho = 2.7\,g/cm^3$, $\hbar\bar{\omega} = 166\,eV$.)

N = electron density ; $\xi = 1$.

Also, $dE = m v\, dv$ so $dx = \frac{m v\, dv}{dE/dx}$

$$R = \int_{v_0}^{0} \frac{m v\, dv \cdot m v^2}{-4\pi N e^4 \ln\left(\frac{5 m v^2}{\hbar\bar{\omega}}\right)}$$

Let $u = \frac{m v^2}{\hbar\bar{\omega}} \Rightarrow du = \frac{2 m v\, dv}{\hbar\bar{\omega}}$

$$R = \frac{-1}{4\pi N e^4} \frac{\hbar\bar{\omega}}{2} \int_{2W_0/\hbar\bar{\omega}}^{0} du \frac{\hbar\bar{\omega}\, u}{\ln(u)}$$

$$R(W_0) = \frac{(\hbar\bar{\omega})^2}{8\pi N e^4} \int_0^{2W_0/\hbar\bar{\omega}} \frac{u\, du}{\ln(u)}$$

Now use Mathematica.

In[39]:=
```
Remove["Global`*"]
I0 = Integrate[u / Log[u], u]
I1 = I0 /. u → 0
I2 = I0 /. u → umax
```

Out[40]= ExpIntegralEi[2 Log[u]]

Out[41]= 0

Out[42]= ExpIntegralEi[2 Log[umax]]

In[96]:=
```
(* Parameters *)
{Z, MAl, ρ, hbω} = {13, 27 * Ug / Umol, 2.7 * Ug / Ucm^3, 166 * UeV}
Umol = 6.02*^23
Ne = Z * ρ / MAl
ee = 4.8*^-10 * Sqrt[Ug * Ucm^3 / Us^2]
K = hbω^2 / (8 * Pi * Ne * ee^4)
K = K // FullSimplify
UeV = (1.6*^-19) * (1.0*^7 * Ug * Ucm^2 / Us^2)
K = K // FullSimplify
```

Out[96]= $\left\{13,\ 4.48505\times10^{-23}\ \mathrm{Ug},\ \frac{2.7\ \mathrm{Ug}}{\mathrm{Ucm}^3},\ \frac{2.656\times10^{-10}\ \mathrm{Ucm}^2\ \mathrm{Ug}}{\mathrm{Us}^2}\right\}$

Out[97]= 6.02×10^{23}

Out[98]= $\frac{7.826\times10^{23}}{\mathrm{Ucm}^3}$

Out[99]= $4.8\times10^{-10}\sqrt{\frac{\mathrm{Ucm}^3\ \mathrm{Ug}}{\mathrm{Us}^2}}$

Out[100]= $6.75634\times10^{-8}\ \mathrm{Ucm}$

Out[101]= $6.75634\times10^{-8}\ \mathrm{Ucm}$

Out[102]= $\frac{1.6\times10^{-12}\ \mathrm{Ucm}^2\ \mathrm{Ug}}{\mathrm{Us}^2}$

Out[103]= $6.75634\times10^{-8}\ \mathrm{Ucm}$

In[104]:=
```
α = 2 * (1000 * UeV / hbω)
R[ξ_] = K * I2 /. {umax → α * ξ} (* ξ = W0 in KeV *)
Print["For W0 = 10 keV, calculated R = ", R[10]]
Print["For W0 = 100 keV, calculated R = ", R[100]]
Plot[R[ξ] / Ucm, {ξ, 10, 100},
 Frame → True, FrameLabel → {"W₀ [keV]", "R(W₀) [cm]"}]
```

Out[104]= 12.0482

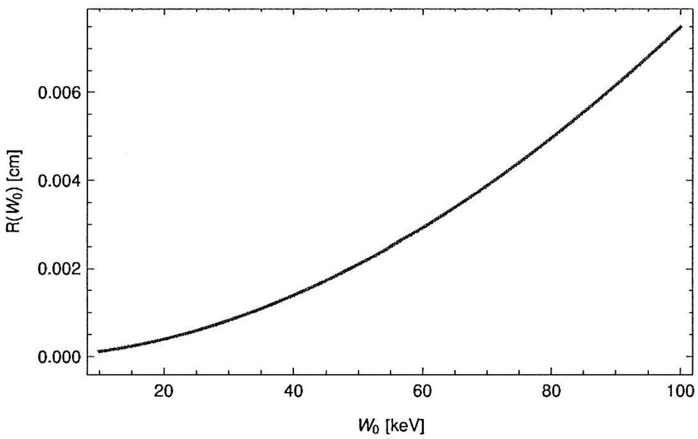
Out[105]= $6.75634\times10^{-8}\ \mathrm{Ucm}\ \mathrm{ExpIntegralEi}[2\ \mathrm{Log}[12.0482\ \xi]]$

6.2.3

For W0 = 10 keV, calculated R = 0.000116628 Ucm

For W0 = 100 keV, calculated R = 0.00748974 Ucm

Out[108]=



Problem 6-3: Exercise 8.6.1

Green's function for the one-dimensional wave function

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(xt; x't') = -4\pi\,\delta(x-x')\,\delta(t-t')$$

Solve by Fourier analysis. Write

$$G(xt; x't') = \int\frac{dk}{2\pi}\int\frac{d\omega}{2\pi}\, e^{ik(x-x')}e^{-i\omega(t-t')}\, G(k\omega)$$

The differential equation $\Rightarrow$

$$(-k^2 + \omega^2/c^2)\, G(k,\omega) = -4\pi$$

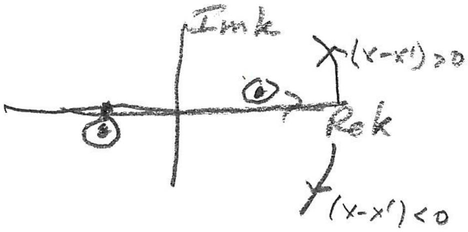
For the retarded time boundary condition,

$$G(k,\omega) = \frac{4\pi}{k^2 - \omega^2/c^2} = \frac{4\pi c^2}{[c^2k^2 - (\omega + i\epsilon)^2]}$$

$\epsilon \to 0+$

Now do the Fourier integrals.

The k integral $\quad J(\omega) = \int\frac{dk}{2\pi}\,\frac{e^{ik(x-x')}}{[c^2k^2 - (\omega+i\epsilon)^2]}$



$\exists$ poles at $ck = \pm(\omega + i\epsilon) = \begin{cases}\omega + i\epsilon \\ -\omega - i\epsilon\end{cases}$

If $x - x' > 0$, close the contour above;

$$J(\omega) = \frac{1}{2\pi}\,\frac{2\pi i}{c}\,\frac{e^{i\omega(x-x')/c}}{2(\omega+i\epsilon)} = \frac{i}{2c}\,\frac{e^{i\omega(x-x')/c}}{(\omega+i\epsilon)}$$

If $x - x' < 0$, close the contour below;

$$J(\omega) = \frac{1}{2\pi}(-2\pi i)\frac{1}{c}\,\frac{e^{-i\omega(x-x')/c}}{-2(\omega+i\epsilon)} = \frac{i}{2c}\,\frac{e^{-i\omega(x-x')/c}}{\omega+i\epsilon}$$

Result

$$J(\omega) = \frac{i}{2c}\left\{\frac{\theta(x-x')}{\omega+i\epsilon}\, e^{i\omega(x-x')/c} + \frac{\theta(x'-x)}{\omega+i\epsilon}\, e^{-i\omega(x-x')/c}\right\}$$

$$J(\omega) = \frac{i}{2c}\,\frac{1}{\omega+i\epsilon}\, e^{i\omega|x-x'|/c}$$

The ω integral

$$G(xt\,;\,x't') = 4\pi c^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')}\, J(\omega)$$

$$= 4\pi c^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega[t-t'-|x-x'|/c]}\, \frac{i}{2c}\,\frac{1}{\omega+i\epsilon}$$

If $t-t'-|x-x'|/c < 0$, close the contour above

$$e^{-i\omega(-X)} = e^{i\omega X} = 0 \text{ at } \omega = +iR\,;$$

then $G = 0$

If $t-t'-|x-x'|/c > 0$, close the contour below

then

$$G = 4\pi c^2 \left(\frac{i}{2c}\right)\frac{1}{2\pi}(-2\pi i)\; e^{-\epsilon[\ldots]} \xrightarrow[\epsilon\to 0]{} c\cdot 2\pi$$

Final Result

$$G(xt;x't') = 2\pi c\; \Theta(t-t'-|x-x'|/c)$$

Problem 6-4 Exercise 8.6.4

(a) Let $G(\vec{x}t; \vec{x}'t') = \frac{1}{R}\delta(\tau - R/c)$ where $R = |\vec{x} - \vec{x}'|$.

and $\tau = t - t'$.

W.L.O.G., let $\vec{x}' = 0$ and $t' = 0$.

Now calculate

$$\nabla^2 G(\vec{x}t; 00) = \nabla^2\left\{\frac{1}{r}\delta(t - r/c)\right\} \quad \left\{\begin{array}{l} t = \tau \text{ for } t' = 0 \\ r = R \text{ for } \vec{x}' = 0 \end{array}\right.$$

$$\nabla^2(ab) = (\nabla^2 a)b + 2\nabla a \cdot \nabla b + a\nabla^2 b$$

$$\nabla^2\left(\frac{1}{r}\delta(t - r/c)\right) = \nabla^2\left(\frac{1}{r}\right)\delta(t - r/c) + 2\nabla\left(\frac{1}{r}\right)\cdot\nabla\delta(t - r/c) + \frac{1}{r}\nabla^2\delta(t - r/c)$$

$$= -4\pi\delta^3(\vec{x})\,\delta(t - r/c) + 2\left(\frac{-\hat{r}}{r^2}\right)\left(-\frac{1}{c}\right)\delta'(t - r/c)\,\hat{r} +$$

$$+ \frac{1}{r}\nabla\cdot\left[\delta'(t - r/c)\left(-\frac{1}{c}\right)\hat{r}\right] \qquad \boxed{\nabla r = \hat{r}}$$

$$= -4\pi\delta^3(\vec{x})\,\delta(t) + \frac{2}{cr^2}\delta'(t - r/c)$$

$$-\frac{1}{cr}\left[\delta''(t - r/c)\left(-\frac{1}{c}\hat{r}\right)\cdot\hat{r} + \delta'(t - r/c)(\nabla\cdot\hat{r})\right]$$

$$\nabla\cdot\hat{r} = \frac{\partial}{\partial x_i}\left(\frac{x^i}{r}\right) = \frac{3}{r} + x_i\left(\frac{-1}{r^2}\right)\frac{x^i}{r} = \frac{2}{r}$$

$$= -4\pi\delta^3(\vec{x})\,\delta(t) + \frac{2}{cr^2}\delta'(t - r/c)$$

$$+ \frac{1}{c^2 r}\delta''(t - r/c) - \frac{2}{cr^2}\delta'(t - r/c)$$

$$= -4\pi\delta^3(\vec{x})\,\delta(t) + \frac{1}{c^2}\frac{1}{r}\delta''(t - r/c)$$

Also, $\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\frac{1}{r}\delta(t - r/c)\right] = \frac{1}{c^2 r}\delta''(t - r/c)$

Result

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]G(\vec{x}t; 00) = -4\pi\delta^3(\vec{x})\,\delta(t)$$

AS CLAIMED

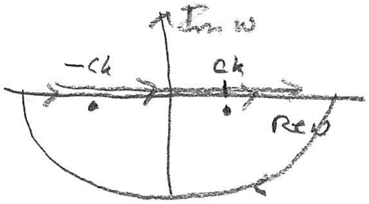
Part b

Use the integral representation

$$G(\vec{x}t;\vec{x}'t') = \int \frac{d^3k}{(2\pi)^3}\int \frac{d\omega}{2\pi}\, e^{i\vec{k}\cdot\vec{R}}\, e^{-i\omega\tau}\, \frac{4\pi c^2}{c^2k^2-(\omega+i\epsilon)^2}$$

where $\vec{R} = \vec{x}-\vec{x}'$ and $\tau = t-t'$.

Do the ω integral.



$$A \equiv \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \frac{e^{-i\omega\tau}\, 4\pi c^2}{(ck-\omega-i\epsilon)(ck+\omega+i\epsilon)}$$

Because $\tau > 0$ we can close the contour in the LH(ω)P $\Rightarrow$

$$A = \frac{4\pi c^2}{2\pi}(-1)(-2\pi i)\left\{ \frac{e^{-i(ck)\tau}}{2ck} + \frac{e^{-i(-ck)\tau}}{-2ck} \right\}$$

$$= \frac{4\pi c^2 i}{2ck}(-2i)\sin(ck\tau) = \frac{4\pi c}{k}\sin(ck\tau)$$

$$\lim_{\tau\to 0+} G(\vec{x},t;\vec{x}',t') = 0 \quad \text{because } \sin(ck\tau)\to 0.$$

$$\frac{\partial G}{\partial t'} = -\frac{\partial G}{\partial \tau} = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{R}}\left(-\frac{4\pi c}{k}\right) ck\cos(ck\tau)$$

$$\lim_{\tau\to 0+} = -4\pi c^2 \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{R}} \quad \text{because } \cos(ck\tau)\to 1$$

$$= -4\pi c^2\,\delta^3(\vec{R}),$$ as claimed.