Problem 6-5 : Exercise 11.1

Out[420]=

Exercise 11.1.1.

(a) Using a computer graphics package, investigate the radiation patterns for the center-fed linear antenna. Illustrate the radiation patterns for $kd = \pi$, 3π and 5π.

(b) Prove the statement in the text relating the number of radiation zeros, $2n$, to the kd value for this antenna. In this process, show they are located at

$$\cos\theta = \pm\left(1 - \frac{4\pi m}{kd}\right), \quad m = 1, 2, \ldots, n,$$

where n is the greatest integer less than $kd/(2\pi)$.

Out[421]=

$$\left(\frac{dP}{d\Omega}\right)_{\text{avg}} = \frac{I^2}{2\pi c}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta}\right]^2. \quad (11.36)$$

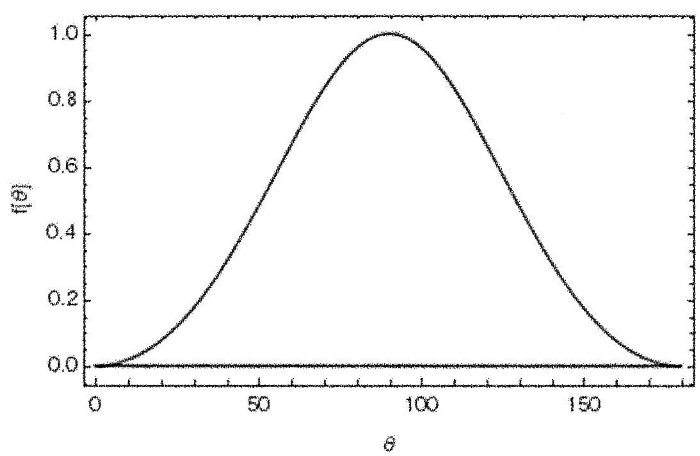
Solution for part (a)

In[2]:=
```
f[a_, θ_] = (Cos[a * Cos[θ]] - Cos[a]) ^2 / Sin[θ] ^2;
```

- The angular distribution for kd = π is shown.

In[3]:=
```
Plot[f[Pi / 2, θ * Pi / 180], {θ, 0, 180},
  Frame → True, FrameLabel → {"θ", "f[θ]"},
  Epilog → Line[{{0, 0}, {180, 0}}]] // Rasterize
```

Out[3]=

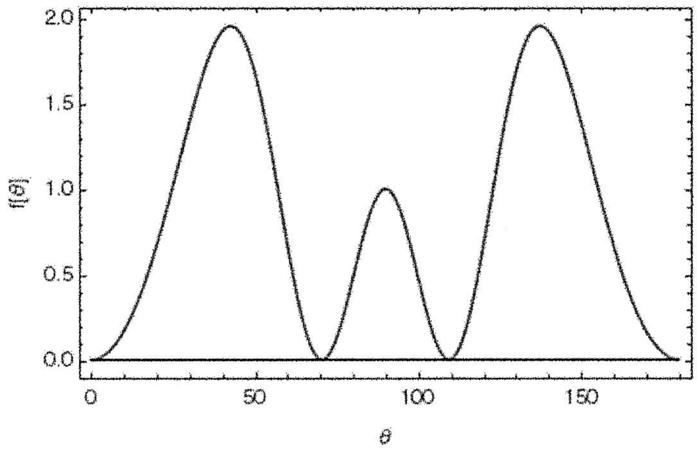


- The angular distribution for kd = 3π is shown.

6-5.3

In[4]:=
```
Plot[f[3 * Pi / 2, θ * Pi / 180], {θ, 0, 180},
  Frame → True, FrameLabel → {"θ", "f[θ]"},
  Epilog → Line[{{0, 0}, {180, 0}}]] // Rasterize
```

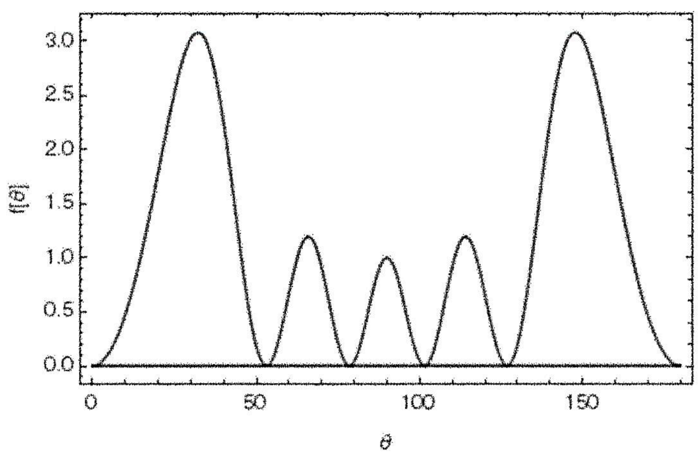
Out[4]=



- The angular distribution for kd = 5π is shown.

In[5]:=
```
Plot[f[5 * Pi / 2, θ * Pi / 180], {θ, 0, 180},
  Frame → True, FrameLabel → {"θ", "f[θ]"},
  Epilog → Line[{{0, 0}, {180, 0}}]] // Rasterize
```

Out[5]=



Solution for part (b)

We see from the plots:

- kd = 1 π has 0 radiation zeroes,
- kd = 3 π has 2 radiation zeroes,
- kd = 5π has 4 radiation zeroes.

So evidently, kd = (2n+1)π has 2n radiation zeroes.

Where are the radiation zeroes?

dP/dΩ = 0 implies $\cos(\, kd/2 \cos(\theta)\,) = \pm \cos(\, kd/2\,)$.

∴ $(kd/2)\cos(\theta) = kd/2 + \pi m$, where m = 1, 2, 3 ,4, ...

I.e.,

$$\cos(\theta_m) = 1 - \frac{2\pi m}{kd} \quad \text{where } m = 1, 2, \ldots, m_{max}$$

For example, suppose kd = 3π.

Then the radiation zeroes are at

m = 1 $\Longrightarrow \cos(\theta_1) = 1 - \frac{2}{3} = \frac{1}{3}$; θ_1 = 70.5 degrees

m = 2 $\Longrightarrow \cos(\theta_2) = 1 - \frac{4}{3} = -\frac{1}{3}$; θ_2 = 109.5 degrees

Or, suppose kd = 5π.

Then the radiation zeroes are at

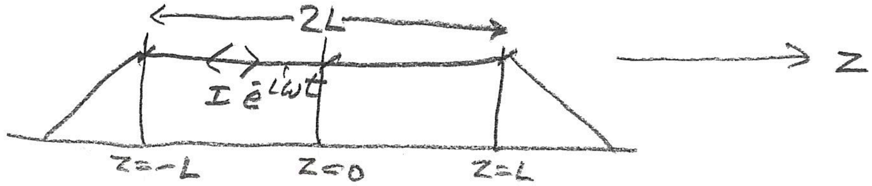
m = 1 $\Longrightarrow \cos(\theta_1) = \frac{3}{5}$; θ_1 = 53 degrees

m = 2 $\Longrightarrow \cos(\theta_2) = \frac{1}{5}$; θ_2 = 78 degrees

m = 3 $\Longrightarrow \cos(\theta_3) = -\frac{1}{5}$; θ_1 = 102 degrees

m = 4 $\Longrightarrow \cos(\theta_4) = -\frac{1}{5}$; θ_2 = 127 degrees

Problem 6-6 (Exercise 11.1.3)



(a) $\vec{J}(\vec{x}', t') = \vec{J}(\vec{x}') e^{-i\omega t'}$

$$\vec{J}(\vec{x}') = I\ \delta(x')\ \delta(y')\ \hat{e}_3 \quad \text{for } -L \le z' \le L.$$

Use

$$\left(\frac{dP}{d\Omega}\right)_{avg} = \frac{k^2}{8\pi c} \left| \hat{n} \times \int d^3x'\ \vec{J}(\vec{x}')\ e^{-ik\hat{n}\cdot\vec{x}'} \right| \qquad (11.23)$$

$k = \omega/c$

= angular distribution of radiated power

$$\hat{n} = \hat{e}_x \sin\theta\cos\phi + \hat{e}_y \sin\theta\sin\phi + \hat{e}_z \cos\theta$$

$$\hat{n}\times\vec{g} = \hat{n}\times\hat{e}_3 \int d^3x'\ I\ \delta(x')\ \delta(y')\ e^{-ik\hat{n}\cdot\vec{x}'}$$

$$= \left(-\hat{e}_2 \sin\theta\cos\phi + \hat{e}_1 \sin\theta\sin\phi\right) I \int_{-L}^{L} dz'\ e^{-ikz'\cos\theta}$$

$$= \hat{e}_\phi\ I\ \frac{1}{-ik\cos\theta}\left\{ e^{-ikL\cos\theta} - e^{ikL\cos\theta} \right\}$$

$\Rightarrow -2i\sin(kL\cos\theta)$

$$= \hat{e}_\phi\ \frac{2I}{k\cos\theta}\sin(kL\cos\theta)$$

$$\therefore \left(\frac{dP}{d\Omega}\right)_{avg} = \frac{k^2}{8\pi c}\frac{4I^2}{k^2\cos^2\theta}\sin^2(kL\cos\theta) = \frac{I^2}{2\pi c}\sin^2(kL\cos\theta)\tan^2(\theta)$$

Part (b)

$$P_{avg} = \int d\Omega \left.\frac{dP}{d\Omega}\right|_{avg} = \frac{1}{2} R' I^2 \cdot (2L)$$

($R' = \frac{dR}{dL}$ = resistance per unit length)

$$P_{avg} = \int_0^{\pi} \sin\theta\, d\theta \int_0^{\pi} d\phi \; \frac{I^2}{2\pi c} \sin^2(kL\cos\theta) \tan^2\theta$$

Let $u = \cos\theta$

$$= \frac{I^2}{c} \int_{-1}^{1} du \, \sin^2(kLu) \, \frac{1-u^2}{u^2}$$

Thus the resistance per unit length is

$$R' = \frac{1}{cL} \int_{-1}^{1} du \, \sin^2(kLu) \, \frac{1-u^2}{u^2}$$

and we want this in the limit $L \to \infty$.

Recall this familiar result

$$\frac{\sin^2(au)}{a u^2} \xrightarrow[a \to \infty]{} \pi \delta(u)$$

Thus $R' = \frac{\pi k}{c} = \frac{\pi \omega}{c^2}$. (in gaussian units)

For $\omega = 2\pi f$ and $f = 60$ Hz, $R' = 1.318 \times 10^{-18} \frac{\text{sec}}{\text{cm}^2}$

Convert to SI units, $1 \text{ ohm} = \frac{10^{-11}}{(2.998)^2}$ sec/cm

Thus $R' = 1.184 \times 10^{-4}$ Ω/m