Problem 7-1 = Exercise 11.2.1 (4 points)

- Electric dipole radiation

$\rho(\vec{x}', t') = -\vec{p}(t') \cdot \vec{\nabla}' \delta^3(\vec{x}')$

Use the continuity equation to get the current density.

$$\nabla' \cdot \vec{J}(\vec{x}', t') = -\frac{\partial}{\partial t'} \rho(\vec{x}', t') = +\dot{\vec{p}}(t') \cdot \nabla' \delta^3(\vec{x}')$$

$$= \nabla' \cdot [\dot{\vec{p}}(t') \delta^3(\vec{x}')]$$

$$\vec{J}(\vec{x}', t') = \dot{\vec{p}}(t') \delta^3(\vec{x}')$$ ↙ 2 points

Now calculate $\frac{dP}{d\Omega} = \frac{1}{4\pi c}\left[\hat{n} \times \frac{1}{c}\int d^3x' \, \frac{\partial \vec{J}(\vec{x}', t_r)}{\partial t}\right]^2$ (11.54)

$$\simeq \frac{1}{4\pi c^3}\left[\hat{n} \times \ddot{\vec{p}}(t_0)\right]^2 \text{ as claimed.}$$

$$\begin{cases} t_r = t - r/c + \hat{n} \cdot \vec{x}'/c \\ t_0 = t - r/c \end{cases}$$

- Magnetic dipole radiation

$$\vec{J}(\vec{x}', t') = -c\, \vec{m}(t') \times \vec{\nabla}' \delta^3(\vec{x}')$$

$$\frac{\partial \vec{J}(\vec{x}', t')}{\partial t'} = -c\, \dot{\vec{m}}(t') \times \nabla' \delta^3(\vec{x}')$$

$$\int d^3x' \, \frac{\partial \vec{J}(\vec{x}', t_r)}{\partial t} = -c\int d^3x' \, \dot{\vec{m}}(t_r) \times \nabla' \delta^3(\vec{x}') = +c\int d^3x' \, \vec{\nabla}' \times \dot{\vec{m}}(t_r)\, \delta^3(\vec{x}')$$

$$= c\int d^3x' \, \ddot{\vec{m}}(t_r) \times \nabla' t_r \, (-1)\, \delta^3(\vec{x}')$$ $\quad \nabla' t_r = \hat{n}/c$

$$= c \, \frac{\hat{n}}{c} \times \ddot{\vec{m}}(t_0)$$

$$\frac{dP}{d\Omega} = \frac{1}{4\pi c}\left[\hat{n} \times \left(\frac{\hat{n}}{c} \times \ddot{\vec{m}}(t_0)\right)\right]^2$$

$$= \frac{1}{4\pi c^3}\left(\hat{n} \times \ddot{\vec{m}}(t_0)\right)^2 \text{ as claimed}$$

$$[\hat{n}(\hat{n} \cdot \ddot{\vec{m}}) - \ddot{\vec{m}}]^2$$
$$= (\hat{n} \cdot \ddot{\vec{m}})^2 + \ddot{\vec{m}}^2 - 2(\hat{n} \cdot \ddot{\vec{m}})^2$$
$$= \ddot{\vec{m}}^2 - (\hat{n} \cdot \ddot{\vec{m}})^2$$
$$= (\hat{n} \times \ddot{\vec{m}})^2$$

↖ 2 points

Problem 7-3

(2 points)

Exercise 11.2.4

Start with Eq (11.54) $\Rightarrow$

$$\frac{dP(t)}{d\Omega} = \frac{1}{4\pi c}\left[\hat{n}\times\frac{1}{c}\int d^3x'\,\frac{\partial}{\partial t}\,\vec{J}(\vec{x}', t_r)\right]$$

where $t_r = t - (r - \hat{n}\cdot\vec{x}')/c$

$R = r - \hat{n}\cdot\vec{x}'$

Now put

$$\vec{J}(\vec{x}', t_r) = \sum_n J_n(\vec{x}')\sin(\omega_n t_r - \delta_n(\vec{x}'))$$

$\Rightarrow$

$$\frac{dP(t)}{d\Omega} = \frac{1}{4\pi c}\left[\hat{n}\times\frac{1}{c}\sum_n\int d^3x'\,J_n(\vec{x}')\;\omega_n\cos(\omega_n t_r - \delta_n(\vec{x}'))\right]^2$$

$$= \frac{1}{4\pi c^3}\sum_\nu\sum_\mu\int_{d^3x''} d^3x'\,(\hat{n}\times J_\nu(\vec{x}'))\cdot(\hat{n}\times J_\mu(\vec{x}''))\,\omega_\nu\omega_\mu \quad *$$

$$\cos(\omega_\nu t_r - \delta_\nu(\vec{x}'))\cos(\omega_\mu t_r - \delta_\mu(\vec{x}''))$$

Average over a long time T

2 points

$$\frac{1}{T}\int dt\,\cos(\omega_\nu t - \omega_\nu R/c - \delta_\nu(\vec{x}'))\cos(\omega_\mu t - \omega_\mu R/c - \delta_\mu(\vec{x}''))$$

$$\xrightarrow[T\to\infty]{} \frac{1}{2}\delta_{\mu\nu}\cos(\omega_\nu R/c + \delta_\nu(\vec{x}'))\cos(\omega_\nu R/c + \delta_\nu(\vec{x}'')) \quad *$$
$$+\frac{1}{2}\delta_{\mu\nu}\sin(\omega_\nu R + \delta_\nu(\vec{x}'))\sin(\omega_\nu R + \delta_\nu(\vec{x}''))$$

$$\therefore \left(\frac{dP}{d\Omega}\right)_{average} = \frac{1}{8\pi c^3}\sum_\nu \omega_\nu^2\left|\int d^3x'\,\hat{n}\times\vec{J}_\nu(\vec{x}')\,e^{i\omega_\nu R/c}\,e^{i\delta_\nu(\vec{x}')}\right|^2$$

$$= \frac{1}{8\pi c}\sum_\nu k_\nu^2\left|\hat{n}\times\int d^3x'\,J_\nu(\vec{x}')\,e^{-ik_\nu\hat{n}\cdot\vec{x}'}\,e^{i\delta_\nu(\vec{x}')}\right|^2$$

$$k_\nu = \omega_\nu/c$$

Problem 7-4 (2 points)

Exercise 11.3.1

Consider this time dependent function,

$$f(t) = 1/\sqrt{2\pi} \int_{-\infty}^{\infty} d\omega\, \tilde{f}(\omega)\, e^{-i\omega t}\,.$$

The inverse transform is

$$\tilde{f}(\omega) = 1/\sqrt{2\pi} \int_{-\infty}^{\infty} dt\, f(\omega)\, e^{i\omega t}\,.$$

Now calculate

$$\int_{-\infty}^{\infty} dt \mid f(t) \mid^2$$

$$= 1/(2\pi) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt\, d\omega_1\, d\omega_2\, \tilde{f}(\omega_1)\, \tilde{f}(\omega_2)^*\, e^{i\omega 1 t} \quad e^{-i\omega 2 t}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega_1\, d\omega_2\, \tilde{f}(\omega_1)\, \tilde{f}(\omega_2)^*\, \delta(\omega_1 - \omega_2)$$

$$= \int_{-\infty}^{\infty} d\omega_1\, \tilde{f}(\omega_1)\, \tilde{f}(\omega_1)^*$$

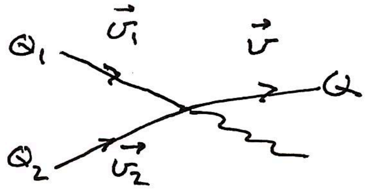
$$= \int_{-\infty}^{\infty} d\omega \mid \tilde{f}(\omega) \mid^2 ;$$

and that proves Parseval's theorem.

2 points

Problem 7-5

Exercise 11.3.2 (4 points)



Momentum conservation $\vec{p}_1 + \vec{p}_2 = \vec{P}$ (neglect $\vec{p}_\gamma$) where

$$m_1 \gamma_1 \vec{v}_1 = \vec{p}_1, \quad m_2 \gamma_2 \vec{v}_2 = \vec{p}_2, \quad M\gamma \vec{v} = \vec{P}.$$

(a) The current density

$$\vec{J}(\vec{x},t) = \begin{cases} Q_1 \vec{v}_1 \delta^3(\vec{x} - \vec{v}_1 t) + Q_2 \vec{v}_2 \delta^3(\vec{x} - \vec{v}_2 t) & \text{for } t<0 \\ Q\vec{v}\, \delta^3(\vec{x} - \vec{v}t) & \text{for } t>0. \end{cases}$$

Then following the derivation of Eqs. (11.79) and (11.80)

2 points

$$\frac{d^2E(\omega)}{d\omega\, d\Omega} = \frac{1}{4\pi^2 c^3} \left| \hat{n} \times \vec{a} \right|^2 \text{ where } \vec{a} = \frac{Q\vec{v}}{1-\hat{n}\cdot\vec{v}/c} - \frac{Q_1 \vec{v}_1}{1-\hat{n}\cdot\vec{v}_1/c} - \frac{Q_2 \vec{v}_2}{1-\hat{n}\cdot\vec{v}_2/c}$$

(b) Derive the condition for an "amplitude zero".

$$\vec{a} = \frac{Q\vec{P}}{M\gamma(1-\hat{n}\cdot\vec{v}/c)} - \frac{Q_1 \vec{p}_1}{m_1\gamma_1(1-\hat{n}\cdot\vec{v}_1/c)} - \frac{Q_2 \vec{p}_2}{m_2\gamma_2(1-\hat{n}\cdot\vec{v}_2/c)}$$

$$= \frac{Q(\vec{p}_1+\vec{p}_2)}{M\gamma(1-\hat{n}\cdot\vec{v}/c)} - \frac{Q_1 \vec{p}_1}{m_1\gamma_1(1-\hat{n}\cdot\vec{v}_1/c)} - \frac{Q_2 \vec{p}_2}{m_2\gamma_2(1-\hat{n}\cdot\vec{v}_2/c)}$$

$$= \vec{p}_1 \left[\frac{Q}{M\gamma(1-\hat{n}\cdot\vec{v}/c)} - \frac{Q_1}{m_1\gamma_1(1-\hat{n}\cdot\vec{v}_1/c)} \right] + \vec{p}_2 \left[\frac{Q}{M\gamma(1-\hat{n}\cdot\vec{v}/c)} - \frac{Q_2}{m_2\gamma_2(1-\hat{n}\cdot\vec{v}_2/c)} \right]$$

$$= 0$$

So the condition is

2 points

$$\frac{Q}{M\gamma(1-\hat{n}\cdot\vec{v}/c)} = \frac{Q_1}{m_1\gamma_1(1-\hat{n}\cdot\vec{v}_1/c)} = \frac{Q_2}{m_2\gamma_2(1-\hat{n}\cdot\vec{v}_2/c)}$$

(not the same as the statement given in the book) *

Problem T-6

Exercise 11.5.3 (6 points)

(a) From Exercise 11.2.2,

$$\frac{dP}{d\Omega} = K\left[(J_0(x)+J_2(x))^2 \sin^2\gamma - 4J_0(x)J_2(x)\cos^2(\phi-\omega t_0)\right]$$

where $K = \frac{\pi \rho_0^2 \omega^4 a^8}{4c^3}$ and $\cos\gamma = -\sin\theta \sin(\phi-\omega t_0)$.

and $x = ka\sin\theta$. Now take the limit $ka \ll 1$.

$$\Rightarrow \quad \frac{dP}{d\Omega} = K\left[\sin^2\gamma\right] \qquad J_0(x)\to 1,\ J_2(x)\to 0$$

← 2 points

(b) According to equation (11.93)

$$\frac{dP}{d\Omega} = \frac{\ddot{\vec{p}}(t_0)^2}{4\pi c^3}\sin^2\theta(t_0) \text{ where } \vec{p}(t_0) = \int d^3x'\,\vec{x}'\,\rho(\vec{x}',t_0)$$

$$= \int d^3x'\,\vec{x}'\,\rho_0 a\,\delta(r'-a)\,\delta(\cos\theta')\sin(\phi'-\omega t_0) \qquad (t_0 = t-r/c)$$

$$= \int_0^{2\pi} d\phi'\left[-\hat{e}_x\sin\phi' + \hat{e}_y\cos\phi'\right]a\,\rho_0 a^3\sin(\phi'-\omega t_0)$$

$$\{\sin\phi'\cos\omega t_0 - \cos\phi'\sin\omega t_0\}$$

$$= \pi\rho_0 a^4\left[-\hat{e}_x\cos\omega t_0 - \hat{e}_y\sin\omega t_0\right] = -\pi\rho_0 a^4\left[\hat{e}_x\cos\omega t_0 + \hat{e}_y\sin\omega t_0\right]$$

$$\ddot{\vec{p}}(t_0) = \pi\rho_0 a^4\omega^2\left[\hat{e}_x\cos\omega t_0 + \hat{e}_y\sin\omega t_0\right]$$

$$\ddot{\vec{p}}(t_0)^2 = \pi^2\rho_0^2 a^8\omega^4$$

$$\therefore \frac{dP}{d\Omega} = \frac{1}{4\pi c^3}\pi^2\rho_0^2 a^8\omega^4\sin^2\theta(t_0) = K\sin^2\theta \quad \text{which agrees with part (a).}$$

because $\theta(t_0)$ = angle between $\vec{p}(t_0)$ and $\hat{n}$ $(=\gamma)$.

← 2 points

(c) Slowing down by radiation

By conservation of energy, where E = rotational energy of the ring, $= \frac{1}{2} m a^2 \omega^2$,

$$\frac{dE}{dt} = -\int \frac{dP}{d\Omega}\, d\Omega$$

$$= -\int K \sin^2\theta \sin\theta\, d\theta\, d\phi = -K\int_{-1}^{1}(1-u^2)du \int_0^{2\pi} d\phi = -\frac{8\pi}{3} K$$

Here $E = \frac{1}{2} m a^2 \omega^2$ so $\frac{dE}{dt} = m a^2 \omega \frac{d\omega}{dt}$

and $K = \frac{\pi \rho_0^2 \omega^4 a^8}{4c^3}$ from part (a),

$$\therefore \quad \frac{d\omega}{dt} = -\frac{\pi \rho_0^2 a^6}{4mc^3}\,\omega^3$$

$$\frac{d\omega}{\omega^3} = -\frac{\pi \rho_0^2 a^6}{4mc^3}\, dt \Rightarrow -\frac{1}{2\omega^2}\Big|_{\omega_0}^{\omega(t)} = -\frac{\pi \rho_0^2 a^6}{4mc^3}\, t$$

$$\frac{1}{\omega^2(t)} - \frac{1}{\omega_0^2} = \frac{2\pi \rho_0^2 a^6}{4mc^3}\, t$$

$$\frac{1}{2} m a^2 \left[\frac{1}{E(t)} - \frac{1}{E_0}\right] = \frac{2\pi \rho_0^2 a^6}{4mc^3}\, t$$

← 2 points

Result

$$\frac{1}{E(t)} - \frac{1}{E_0} = \frac{4\pi \rho_0^2 a^4}{4 m^2 c^3}\, t = kt$$

where $k = \frac{\pi \rho_0^2 a^4}{m^2 c^3}$