Problem 7-4 (Exercise 11.2.2) part (a)

Some preliminary calculations

Given this charge density,

$\rho\left(\vec{x}', t'\right) = \rho_0\, a\, \delta(r'-a)\, \delta(\cos\theta')\, \sin(\phi' - \omega t')$

The electric dipole moment is

$\vec{p}(t') = \int d^3x' \;\; \vec{x}'\, \rho\left(\vec{x}', t'\right)$

$= \rho 0 \;\; a^3 \int d\phi' \sin(\phi' - \omega t')\{\, e_x\, a \cos\phi' + e_y\, a \sin\phi' \,\}$

$= \rho 0\; a^4\, \pi\, \{\, - e_x \sin(\omega\, t') + e_y \cos(\omega\, t') \,\}$

Thus

$\ddot{\vec{p}}\,(t_0) = -\rho 0\; a^4\, \pi\, \omega^2\, \{\, - e_x \sin(\omega t_0) + e_y \cos(\omega t_0) \,\}$

$|\, \ddot{\vec{p}}\,(t_0)\, | = \rho 0\; a^4\, \pi\, \omega^2$

$\ddot{\vec{p}}\,(t_0) \bullet \hat{n} = -\rho 0\; a^4\, \pi\, \omega^2\, \{\, - \sin\theta \cos\phi \sin(\omega t_0) + \sin\theta \sin\phi \cos(\omega t_0) \,\}$

$\quad = -\rho 0\; a^4\, \pi\, \omega^2 \sin\theta \sin(\phi - \omega t')$

$\cos[\gamma(t_0)] = -\sin\theta \sin(\phi - \omega t_0)$

<u>The exact (i.e., time-dependent) power distribution</u>

We'll calculate the power distribution from Eqs (11.55) and (11.41),

$dP/d\Omega = r^2/(4\pi c)\,(\hat{n} \times \partial\vec{A}/\partial t)^2$ and $\vec{A}(\vec{x},t) \approx 1/(cr) \int d^3x'\, \vec{J}(\vec{x}', t_r)$

where $t_r = t - r/c + \hat{n} \cdot \vec{x}'/c = t_0 + \hat{n} \cdot \vec{x}'/c$; $t_0 = t - r/c$.

The current density is $\vec{J}(\vec{x}',t') = \rho(\vec{x}',t')\,\vec{v} = \rho(\vec{x}',t')\,\omega a\,\hat{e}(\phi')$.

So,

$\vec{A}(\vec{x},t) = 1/(cr) \int d^3x'\, \rho 0\, a^2\, \omega\, \delta(r'-a)\, \delta(\cos\theta')\, \sin(\phi' - \omega t_r)\, \hat{e}(\phi')$

$= \Lambda \int d\phi' \sin(\phi' - \omega t_r)\{-\hat{e}_x \sin\phi' + \hat{e}_y \cos\phi'\}$

where

$\Lambda = (\rho 0\, a^4\, \omega)/(cr)$ and $t_r = t_0 + \hat{n} \cdot \vec{x}'/c$; $t_0 = t - r/c$.

$\hat{n} \cdot \vec{x}' = (\hat{e}_x \sin\theta \cos\phi + \hat{e}_y \sin\theta \sin\phi + \hat{e}_z \cos\theta) \cdot (\hat{e}_x\, a \cos\phi' + \hat{e}_y\, a \sin\phi')$

$= a \sin\theta \cos(\phi - \phi')$

$\vec{A}(\vec{x}, t) = \Lambda \int d\phi' \sin[\phi' - \omega t_0 + k a \sin\theta \cos(\phi-\phi')]\{-\hat{e}_x \sin\phi' + \hat{e}_y \cos\phi'\}$

Note $k = \omega/c$.

Let $k a \sin\theta = \xi$.

The time derivative —

$\dot{\vec{A}} = -\omega \Lambda \int d\phi' \cos[\phi' - \omega t_0 + \xi \cos(\phi-\phi')]\{-\hat{e}_x \sin\phi' + \hat{e}_y \cos\phi'\}$

<u>Important trick: Expand in Bessel functions</u>

Equation (4.11) : $\exp\{i\, \xi \cos(\phi-\phi')\} = \Sigma_m\; i^m\, e^{im(\phi-\phi')}\, J_m(\xi)$

Write $\cos[...] = \mathrm{Re}\, \exp[i(\phi' - \omega t_0)] \exp[i\, \xi \cos(\phi-\phi')]$.

$\dot{\vec{A}} = -\omega \Lambda\, \mathrm{Re} \int d\phi' \exp[i(\phi' - \omega t_0)] \exp[i\xi\cos(\phi-\phi')]\{-\hat{e}_x \sin\phi' + \hat{e}_y \cos\phi'\}$

$$= -\omega \wedge \text{Re}\, \Sigma_m\, i^m\, J_m(\xi) \int d\phi' \;\; \exp[i(\phi' - \omega t_0)]\, \exp[\, i\, m\, (\phi - \phi')]$$

$$\{-\hat{e}_x \sin\phi' + \hat{e}_y \cos\phi' \}$$

$$= -\omega \wedge \text{Re}\, \Sigma_m\, i^m\, J_m(\xi)\, e^{i\, m\, \phi}\, e^{-i\, \omega\, t0}$$

$$\times \int d\phi' \;\; \exp[-\, i\, (m-1)\, \phi']\, \{-\hat{e}_x \sin\phi' + \hat{e}_y \cos\phi' \}$$

Note that the integral over ϕ' is zero unless m = 0 or m = 2.

In[∘]:=
```
m = 0;
{A0 = Integrate[Exp[-I (m - 1) ϕ'] Sin[ϕ'], {ϕ', 0, 2 Pi}],
 B0 = Integrate[Exp[-I (m - 1) ϕ'] Cos[ϕ'], {ϕ', 0, 2 Pi}]}
m = 2;
{A2 = Integrate[Exp[-I (m - 1) ϕ'] Sin[ϕ'], {ϕ', 0, 2 Pi}],
 B2 = Integrate[Exp[-I (m - 1) ϕ'] Cos[ϕ'], {ϕ', 0, 2 Pi}]}
```

Out[∘]= $\{i\, \pi,\, \pi\}$

Out[∘]= $\{-i\, \pi,\, \pi\}$

$$\dot{A} = -\omega \wedge \text{Re} \Big\{ J_0(\xi)\, e^{-i\omega t0}\, [\, -e_x\, A0 + e_y\, B0\,]$$

$$-\, J_2(\xi)\, e^{i(2\,\phi - \omega\, t_0)}\, [\, -e_x\, A2 + e_y\, B2\,] \;\Big\}$$

$$\dot{A} = -\omega \wedge \text{Re} \Big\{ J_0(\xi)\, e^{-i\omega t0}\, [\, -e_x\, i\pi + e_y\, \pi]$$

$$-\, J_2(\xi)\, e^{i(2\,\phi - \omega\, t_0)}\, [\, -e_x\, (-i\pi) + e_y\, \pi] \;\Big\}$$

$$\dot{A} = -\omega \wedge \pi \{\, J_0(\xi)\, [\, -e_x \sin(\omega t_0) \; + e_y \cos(\omega t_0)\,]$$

$$-\, J_2(\xi)\, [\, -e_x \sin(2\,\phi - \omega t_0) \; + e_y \cos(2\,\phi - \omega t_0)\,] \;\}$$

<u>Next, calculate $\hat{n} \times \dot{A}$</u>

$$\hat{n} = e_x \sin\theta \cos\phi + e_y \sin\theta \sin\phi + e_z \cos\theta$$

$$\hat{n} \times e_x = -\, e_z \sin\theta \sin\phi + e_y \cos\theta$$

$$\hat{n} \times e_y = +\, e_z \sin\theta \cos\phi - e_x \cos\theta$$

$$\hat{n} \times \dot{A} = -\omega \wedge \pi \{$$

$$J_0(\xi)[\, -(\, -e_z \sin\theta \sin\phi + e_y \cos\theta) \sin(\omega t_0)$$

$$+(e_z \sin\theta \cos\phi - e_x \cos\theta) \cos(\omega t_0)\,]$$

$-J_2(\xi)[\ -(\ -e_z \sin\theta \sin\phi\ + e_y \cos\theta) \sin(2\,\phi - \omega t_0)$

$+(e_z \sin\theta \cos\phi - e_x \cos\theta) \cos(2\,\phi - \omega t_0)\ ]\ \}$

$\hat{n} \times \dot{A} = -\omega\,\Lambda\,\pi\,\{$

$J_0(\xi)\ [\ e_z \sin\theta \cos(\phi - \omega t_0)\ +\ \cos\theta\,(-e_x \cos(\omega t_0) - e_y \sin(\omega t_0)\ )\,]$

$-J_2(\xi)\ [\ e_z \sin\theta \cos(\phi - \omega t_0)\ + \cos\theta\ (-e_x \cos(2\,\phi - \omega t_0) - e_y \sin(2\,\phi - \omega t_0)\ )$

$]\ \}$

Simplifications

term#1 = $\hat{e}_z\ (-\omega\Lambda\pi)\ (J_0 - J_2) \sin\theta \cos(\phi - \omega t_0)$

term#2 = $\hat{e}_x \cos\theta\ (-\omega\Lambda\pi)\ \{- J_0 \cos(\omega t_0) + J_2 \cos(2\,\phi - \omega t_0)\}$

term#3 = $\hat{e}_y \cos\theta\ (-\omega\Lambda\pi)\ \{- J_0 \sin(\omega t_0) + J_2 \sin(2\,\phi - \omega t_0)\}$

Now calculate $(\hat{n} \times \dot{A})^2$

$(\hat{n} \times \dot{A})^2 = (t1)^2 + (t2)^2 + (t3)^2$

The rest is just algebra.
It is not too difficult to calculate $t1^2 + t2^2 + t3^2$.
It makes it easier to use Mathematica.
The final result is

$$(\hat{n} \times \dot{A})^2 = (\omega\Lambda\pi)^2\ \{\ (\ J_0(\xi) + J_2(\xi)\ )^2 \sin^2\gamma - 4\ J_0(\xi)\ J_2(\xi) \cos^2(\phi - \omega\ t_0)\ \}$$

Recall $\Lambda = (\rho_0\ a^4\ \omega)\ /\ (cr)$.

Thus,

$dP/d\Omega = r^2/(4\pi c)\ (\ \hat{n} \times \dot{A}\ )^2$

$= \pi\,\rho_0^2\ \omega^4\ a^8\ /\ \left(4\ c^3\right)\ \{\ (\ J_0(\xi) + J_2(\xi)\ )^2 \sin^2\gamma - 4\ J_0(\xi)\ J_2(\xi) \cos^2(\phi - \omega\ t_0)\ \}$

Part (b)

Now calculate the time-averaged radiation rate.

We'll have $\langle \cos^2(\phi - \omega\ t_0) \rangle$ and $\langle \sin^2(\phi - \omega\ t_0) \rangle$, which are equal to 1/2.

Also, $\langle \sin^2 \gamma \rangle = 1 - \langle \cos^2 \gamma \rangle = 1 - (1/2) \sin^2 \theta = (1/2)\,(1 + \cos^2 \theta)$.

So the result is

$$\langle dP/d\Omega \rangle = \pi \rho_0^2\, \omega^4\, a^8 \,/\left(4\, c^3\right) \cdot \{ (J_0 + J_2)^2/2 * \left(1 + \cos^2 \theta\right) - 2\, J_0\, J_2 \}$$

$$= \pi \rho_0^2\, \omega^4\, a^8 \,/\left(8\, c^3\right) \cdot \{ (J_0 - J_2)^2 + \cos^2 \theta \, (J_0 + J_2)^2 \}$$