Homework Assignment #8

Problem 8-1 ; Exercise 11.5.5 ; [4 points]

In[700]:=
```
dir0 = "/Users/OurMacBookAir/Documents";
dir1 = "/Teaching.2018.current/chapter11.current/HA8";
Import[StringJoin[dir0, dir1, "/Ex1155a.png"], "PNG"]
Import[StringJoin[dir0, dir1, "/Ex1155b.png"], "PNG"]
```

Out[702]=

Exercise 11.5.5. Given the expression for instantaneous angular power,

$$\frac{dP(t)}{d\Omega} = \frac{1}{4\pi c}\left[\hat{n} \times \int d^3x' \, \frac{1}{c}\frac{\partial \vec{J}(\vec{x}\,', t_r)}{\partial t}\right]^2,$$

Out[703]=

where $t_r = t_0 + \hat{n} \cdot \vec{x}\,'/c(t_0 = t - r/c)$, show that for a spherically symmetric radial current,

$$\vec{J}(\vec{x}\,', t) = \hat{n}' f(r', t), \ (r' = |\vec{x}\,'|)$$

there is in fact no radiation, even though charges are being accelerated. (This is the "breathing mode" mentioned in

Consider the integral

$\vec{I} = \int d^3 x' \ \vec{J}\left(\vec{x}\,', t_r\right)$

As far as the angles (θ', ϕ') are concerned $J\left(\vec{x}\,', t_r\right)$ is only a function of $\hat{n}\cdot\vec{x}'$. Therefore the angular part of $\vec{I}$ may be written as

$\vec{I}_\alpha = \int \ d\Omega' \ \hat{n}' \ g(\hat{n}\cdot\vec{x}')$

Since the integral is over all directions, there is no loss of generality to take the z-axis in the direction of $\hat{n}$. Then,

$\vec{I}_\alpha = \int \ \sin\theta' \ d\theta' \ d\phi' \ \hat{n}' \ g(r' \cos\theta')$

Now

$\int d\phi' \ \hat{n}' = 2\pi \ \hat{e}_z \cos\theta'$

so

$\vec{I}_\alpha = 2\pi \ \hat{e}_z \int \ \sin\theta' \ d\theta' \cos\theta' \ g(r' \cos\theta')$

That is, $\vec{I}_\alpha$ is in the $\hat{n}$ ($= \hat{e}_z$) direction.

↖2 points for the proof

Therefore $dP/d\Omega = 0$ because $\hat{n} \times \hat{n} = 0$.

↖2 points for the result

Problem 8-2 ; Exercise 11.5.6 ; [4 points]

In[704]:= `Import[StringJoin[dir0, dir1, "/Ex1156.png"], "PNG"]`

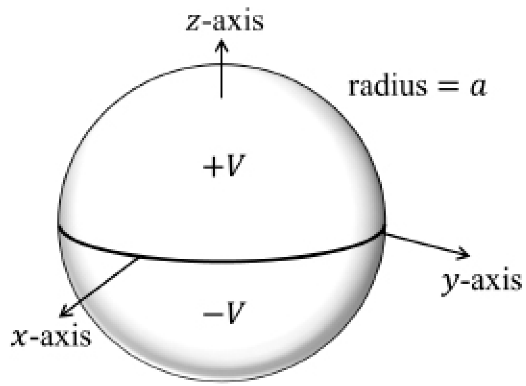

Fig. 11.16 Setup for Exercise 11.5.6.

Out[704]=

Exercise 11.5.6. Consider a thin spherical shell of radius a, slightly separated along its equator, with static equal but opposite potentials on the two halves (see Figure 11.16). The shell is rotated at a constant angular velocity ω about an axis. Find the time-averaged angular power, $(dP/d\Omega)_{avg}$, radiated from the system in the long-wavelength limit when the axis of rotation is:

(a) the z-axis;
(b) the x-axis.

In the long-wavelength limit,

$$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3}\left[\hat{n} \times \ddot{\vec{p}}(t_0)\right]^2$$

where $t_0 = t - r/c$.

The dipole moment when the sphere is not rotating is

$\vec{p} = \int dA\ \vec{x}\ \sigma(\vec{x})$; here $dA = a^2 \sin\theta\ d\theta\ d\phi$;

also, $\vec{x} = \hat{x}\ a \sin\theta \cos\phi + \hat{y}\ a \sin\theta \sin\phi + \hat{z}\ a \cos\theta$.

By symmetry, $\sigma(\vec{x}) = \sigma_0(\theta)$ for $0 \le \theta \leqslant \pi/2$ and $\sigma(\vec{x}) = -\sigma_0(\pi - \theta)$ for $\pi/2 \leqslant \theta \leqslant \pi$; so

$$\vec{p} = 2\pi a^3\ \hat{e}_z\{\int_0^{\pi/2} d\theta \sin\theta \cos\theta\ \sigma_0(\theta) - \int_{\pi/2}^{\pi} d\theta \sin\theta \cos\theta\ \sigma_0(\pi - \theta)\}$$

$$= 2\pi a^3\ \hat{e}_z \int_0^{\pi/2} d\theta \sin\theta \cos\theta\ \sigma_0(\theta) = p_0\ \hat{e}_z$$

Part (a) [2 points]

If the sphere rotates around the z axis, then
$\vec{p}(t) = p_0\ \hat{e}_z$.
So in this case, $\frac{dP}{d\Omega} = 0$ because $\ddot{\vec{p}}(t) = 0$.

Part (b)

If the sphere rotates around the x axis, then
$\vec{p}(t) = p_0\ (\ \hat{e}_z \cos(\omega t) - \hat{e}_y \sin(\omega t)\ )$
So in this case,

$$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3}\ [\hat{n} \times \ddot{\vec{p}}(t_0)]^2$$

$$= \frac{p_0^2\ \omega^4}{4\pi c^3}\ [\ \hat{n} \times\ [\hat{e}_z \cos(\omega t) - \hat{e}_y \sin(\omega t)]\ ]^2$$

$$= \frac{p_0^2\ \omega^4}{8\pi c^3}\ \{\ 1 + \cos^2\phi\ \sin^2\theta\ \}$$

↖ 2 points

```
(* A calculation *)
nh = {Sin[θ] Cos[ϕ], Sin[θ] Sin[ϕ], Cos[θ]};
pdd = {0, -Sin[ωt], Cos[ωt]};
V = Cross[nh, pdd];
ξ = Dot[V, V] // Expand;
ξ = ξ /. {Cos[ωt]^2 → 1/2, Sin[ωt]^2 → 1/2};
ξ = ξ /. {Cos[ωt] * Sin[ωt] → 0};
ξ = ξ /. {Sin[ϕ]^2 → 1 - Cos[ϕ]^2} // Expand;
ξ = ξ /. {Cos[θ]^2 → 1 - Sin[θ]^2} // Expand
```

Out[979]= $\frac{1}{2} + \frac{1}{2} \text{Cos}[\phi]^2\ \text{Sin}[\theta]^2$

Problem 8-3 ; Exercise 11.5.7 ; [4 points]

In[981]:=
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Out[981]=

Exercise 11.5.7.

(a) A pulsar is a rotating neutron star with a strong magnetic field. Often this magnetic field is not oriented along the rotation axis of the star, but rotates at a polar angle θ_r around the star; see Figure 11.17. Given a uniform angular rotation rate of the star, Ω, find an expression for the instantaneous radiation power loss from the lowest order multipole moment. [*Ans.*:

$$P(t) = \frac{2\vec{m}^2\Omega^4}{3c^3}\sin^2\theta_r.]$$

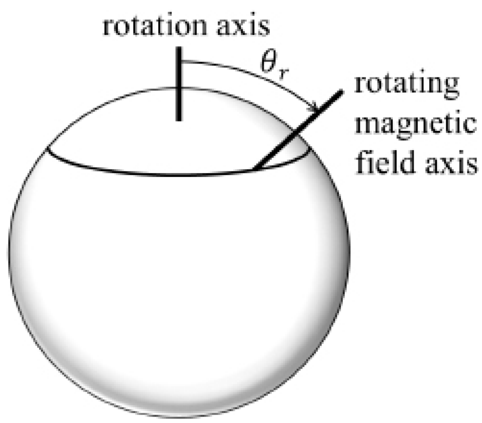

Fig. 11.17 Setup for Exercise 11.5.7.

(b) Define the Earth to be in the direction $\hat{n} = (\sin\theta_E, 0, \cos\theta_E)$. Assuming the Earth and the pulsar star are at rest with respect to one another, find the time dependence of the instantaneous power signal received at the Earth.

Part (a)

The lowest order multipole moment is the magnetic dipole. In the co-rotating frame, the magnetic dipole moment is
$\vec{m} = m_0 \{ \hat{e}_x \sin \Theta_r + \hat{e}_z \cos \Theta_r \}$.
In the frame in which the pulsar rotated with angular rotation rate = Ω is
$\vec{m}(t) = m_0 \{ \hat{e}_x \sin(\Theta_r)\cos(\Omega t) + \hat{e}_y \sin(\Theta_r)\sin(\Omega t) + \hat{e}_z \cos(\Theta_r) \}$.

Then the power radiated per unit solid angle is (114)

$$P(t) = \frac{2}{3\,c^3}\,[\ddot{\vec{m}}(t_0)]^2$$

$$= \frac{2}{3\,c^3}\,m_0^2\,\Omega^4\,\sin^2(\theta_r)$$ ⇐ 2 points

Part (b)

$$\frac{dP}{d\Omega} = \frac{1}{4\,\pi c^3}\left[\,\hat{n}\times\ddot{\vec{m}}\,(t_0)\,\right]^2$$

$$= \frac{1}{4\,\pi c^3}\,m_0^2\,\Omega^4\left\{1-[\,a+b\cos\,(\Omega t)\,]^2\right\}$$ ⇐ 2 points

where $a = \cos\theta E\ \cos\theta r$ and $b = \sin\theta E\ \sin\theta r$

In[1163]:=
```
(* Calculation *)
nh = {Sin[θE], 0, Cos[θE]};
mdd = {Sin[θr] Cos[Ωt], Sin[θr] Sin[Ωt], Cos[θr]};
Dot[mdd, mdd] // Simplify;
Dot[nh, nh] // Simplify;
ξ = 1 - Dot[nh, mdd] ^2
```

Out[1167]= $1-(\text{Cos}[\theta E]\ \text{Cos}[\theta r]+\text{Cos}[\Omega t]\ \text{Sin}[\theta E]\ \text{Sin}[\theta r])^2$

Problem 8-4 ; Exercise 11.5.8 ; 4 points

In[1168]:=
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Out[1168]=

Exercise 11.5.8.

(a) Starting with Equations (11.93)-(11.97), find $dP(t)/d\Omega$, $P(t)$ and $(dP/d\Omega)_{avg}$ for the nonrelativistic motion of a particle with charge e moving in a circle of radius R in a uniform, constant magnetic field, $\vec{B}$.

(b) As the particle radiates, it changes its orbit parameters and looses energy. Show that in nonrelativistic approximation the instantaneous energy $E(t)$ as a function of time is given by

$$E(t) = E_0\,e^{-\lambda t},$$

where E_0 is the energy at $t = 0$. Find the positive constant λ.

Part (a) (2 points)

$$P(t) = \frac{2\, e^2\, \omega^4\, R^2}{3\, c^3}$$

Part (b) (2 points)

$$P(t) = -\frac{dE}{dt} \quad \text{where } E(t) = \frac{1}{2} m\, \omega^2\, R^2.$$

- Nonrelativistic motion of a particle in a magnetic field $B\, \hat{e}_z$:

$$m\, R\, \omega^2 = e\, (R\omega/c)\, B \Longrightarrow \omega = \frac{eB}{mc}$$

- $$\frac{dE}{dt} = -\frac{2\, e^2\, \omega^4\, R^2}{3\, c^3} = -\frac{4\, e^2\, \omega^2}{3\, m\, c^3} \left[\frac{1}{2}\, m\, \omega^2\, R^2\right] = -\lambda\, E$$

where

$$\lambda = \frac{4\, e^2\, \omega^2}{3\, m\, c^3} = \frac{4\, e^4\, B^2}{3\, m^3\, c^5}\ .$$