In[1126]:=
```
dir0 = "/Users/OurMacBookAir/Documents";
dir1 = "/Teaching.2018.current/chapter11.current/HA9";
SetDirectory[StringJoin[dir0, dir1]];
FileNames[];
ex91 = Show[Import["prob.1191.png", "png"], ImageSize → 560];
sol91 = Import["soln.1191.png", "png"];
ex92 = Show[Import["prob.1192.png", "png"], ImageSize → 560];
ex93 = Show[Import["prob.1193.png", "png"], ImageSize → 560];
```

Import: File not found during Import.

Homework Assignment 9 - due Friday October 26

Problem 9-4 = Exercise 11.9.1

In[888]:= ex91

Out[888]=

Exercise 11.9.1. Evaluate $dP_m/d\Omega$ using Equation (11.265) in the nonrelativistic limit $\beta \ll 1$, using the fact that

$$J_m(z) \approx \frac{1}{m!}\left(\frac{z}{2}\right)^m \qquad \text{for } z \ll 1.$$

Show that the $m = 1$ term predominates so that $dP/d\Omega \approx dP_1/d\Omega$, and that your result agrees with the nonrelativistic limit of expression (11.294) (which uses a different definition of θ; see (11.251) and (11.290)).

Problem 9-5 = Exercise 11.9.2

In[889]:= ex92

Out[889]=

Exercise 11.9.2.

(a) Derive Equation (11.294) of this chapter either by a symbolic integration package or explicitly. If explicitly, the following integral formula is from Gradshteyn and Rhyzik, *op. cit.*, 3.661 (4):

$$I_n \equiv \int_0^{2\pi} \frac{dx}{(1 - A\cos x)^{n+1}} = \frac{2\pi}{(1-A^2)^{(n+1)/2}} P_n\left(\frac{1}{\sqrt{1-A^2}}\right)$$

where P_n is the n^{th} Legendre polynomial. Rewrite expression (11.293) for $dP/d\Omega$ in terms of I_4, dI_3/dA, and d^2I_2/dA^2. Then, plug in explicit expressions for the Legendre polynomials and simplify to obtain the result (11.294).

(b) Now using Equation (11.294) show that in the extreme relativistic case ($\beta \approx 1$) the angular power distribution can be approximated as:

$$\frac{dP}{d\Omega} \underset{\beta\approx 1}{\approx} \frac{7}{16}\frac{e^2\omega_0}{2\pi cT}\frac{1}{(\gamma^{-2}+\theta^2)^{5/2}}\left[1 + \frac{5}{7}\frac{\theta^2}{\gamma^{-2}+\theta^2}\right].$$

Part (a)

Start with equation (11.293)

$$\left(\frac{dP}{d\Omega}\right)_{avg} = \frac{e^2\beta^3}{4\pi RT}\int_0^{2\pi} d\phi \, \frac{f(\theta,\phi)}{(1-\beta\cos\theta\cos\phi)^5}$$

where $f = \sin^2\theta + \beta^2\cos^2\theta + \cos^2\theta\cos^2\phi - 2\beta\cos\theta\cos\phi$.

Evaluate the integral with Mathematica.

Mathematica calculations

```
In[891]:= J0 = Integrate[1 / (1 - bb * Cos[ϕ]) ^5, ϕ];
J1 = Integrate[Cos[ϕ] / (1 - bb * Cos[ϕ]) ^5, ϕ];
J2 = Integrate[Cos[ϕ] ^2 / (1 - bb * Cos[ϕ]) ^5, ϕ];
```

In[894]:=
```
{J0, J1, J2} /. {ϕ → 0}
term0 = J0 /. {bb → β * c0};
term0 = term0 * (1 - c0^2 + β^2 * c0^2);
term1 = J1 /. {bb → β * c0};
term1 = term1 * (-2 * β * c0);
term2 = J2 /. {bb → β * c0};
term2 = term2 * c0^2;
```

Out[894]= {0, 0, 0}

In[901]:=
```
term0;
term0a = term0 /. Tan[ϕ / 2] → 0;
term0b = term0 - term0a;
term0a /. ϕ → Pi
term0b = term0b // Expand // Simplify
```

Out[904]= 0

Out[905]=
$$-\frac{\left(8+3\,c0^6\,\beta^4\,\left(-1+\beta^2\right)+8\,c0^2\,\left(-1+4\,\beta^2\right)+3\,c0^4\,\beta^2\,\left(-8+9\,\beta^2\right)\right)\,\mathrm{ArcTanh}\left[\frac{(1+c0\,\beta)\,\mathrm{Tan}\left[\frac{\phi}{2}\right]}{\sqrt{-1+c0^2\,\beta^2}}\right]}{4\,\left(-1+c0^2\,\beta^2\right)^{9/2}}$$

In[911]:=
```
term1;
term1a = term1 /. Tan[ϕ / 2] → 0;
term1b = term1 - term1a;
term1a /. ϕ → Pi;
term1b = term1b // Expand // Simplify
```

Out[915]=
$$\frac{5\,c0^2\,\beta^2\,\left(4+3\,c0^2\,\beta^2\right)\,\mathrm{ArcTanh}\left[\frac{(1+c0\,\beta)\,\mathrm{Tan}\left[\frac{\phi}{2}\right]}{\sqrt{-1+c0^2\,\beta^2}}\right]}{2\,\left(-1+c0^2\,\beta^2\right)^{9/2}}$$

In[916]:=
```
term2;
term2a = term2 /. Tan[ϕ / 2] → 0;
term2b = term2 - term2a;
term2a /. ϕ → Pi
term2b = term2b // Expand // Simplify
```

Out[919]= 0

Out[920]=
$$-\frac{c0^2\,\left(4+27\,c0^2\,\beta^2+4\,c0^4\,\beta^4\right)\,\mathrm{ArcTanh}\left[\frac{(1+c0\,\beta)\,\mathrm{Tan}\left[\frac{\phi}{2}\right]}{\sqrt{-1+c0^2\,\beta^2}}\right]}{4\,\left(-1+c0^2\,\beta^2\right)^{9/2}}$$

In[921]:=
```
all = term0b + term1b + term2b;
all = all // Expand // Simplify
```

Out[922]= $$-\frac{\left(-8+4\,\text{c0}^2+\text{c0}^4\,\left(\beta^2+3\,\beta^4\right)\right)\,\text{ArcTanh}\left[\frac{(1+\text{c0}\,\beta)\,\text{Tan}\left[\frac{\phi}{2}\right]}{\sqrt{-1+\text{c0}^2\,\beta^2}}\right]}{4\,\left(-1+\text{c0}^2\,\beta^2\right)^{7/2}}$$

In[923]:=
```
α = ArcTanh[Sqrt[(1 + β * c0) / (1 - β * c0)] / I * Tan[ϕ / 2]];
α = α /. ϕ → Pi - eps
α = α / (-1) ^ (9 / 2)
factor = -Pi / 2
```

Out[924]= $$-\mathbb{i}\,\text{ArcTan}\left[\sqrt{\frac{1+\text{c0}\,\beta}{1-\text{c0}\,\beta}}\,\text{Tan}\left[\frac{1}{2}\,(-\text{eps}+\pi)\right]\right]$$

Out[925]= $$-\text{ArcTan}\left[\sqrt{\frac{1+\text{c0}\,\beta}{1-\text{c0}\,\beta}}\,\text{Tan}\left[\frac{1}{2}\,(-\text{eps}+\pi)\right]\right]$$

Out[926]= $$-\frac{\pi}{2}$$

In[933]:=
```
all
result = -(-8 + 4 * c0 ^ 2 + c0 ^ 4 * (β ^ 2 + 3 * β ^ 4)) *
  (Pi / 2) / 4 / (1 - β ^ 2 * c0 ^ 2) ^ (7 / 2) * 2
(* note multiplication by 2 *)
result = result * (e ^ 2 * β ^ 3) / (4 Pi * R * T)
```

Out[933]= $$-\frac{\left(-8+4\,\text{c0}^2+\text{c0}^4\,\left(\beta^2+3\,\beta^4\right)\right)\,\text{ArcTanh}\left[\frac{(1+\text{c0}\,\beta)\,\text{Tan}\left[\frac{\phi}{2}\right]}{\sqrt{-1+\text{c0}^2\,\beta^2}}\right]}{4\,\left(-1+\text{c0}^2\,\beta^2\right)^{7/2}}$$

Out[934]= $$\frac{\pi\,\left(8-4\,\text{c0}^2-\text{c0}^4\,\left(\beta^2+3\,\beta^4\right)\right)}{4\,\left(1-\text{c0}^2\,\beta^2\right)^{7/2}}$$

Out[935]= $$\frac{\text{e}^2\,\beta^3\,\left(8-4\,\text{c0}^2-\text{c0}^4\,\left(\beta^2+3\,\beta^4\right)\right)}{16\,\text{R}\,\text{T}\,\left(1-\text{c0}^2\,\beta^2\right)^{7/2}}$$

In[930]:=
```
(* comparison to equation 294 *)
test = (1 - β^2 * c0^2) * (3 - 3 * c0^2 * (1 - β^2)) +
   (5 - c0^2 - 4 * β^2 * c0^4)
test = test // Expand // Simplify
eq294 = test * (e^2 * β^3) / (16 * R * T) / (1 - β^2 * c0^2)^(7 / 2)
```

Out[930]= $5 - c0^2 - 4\, c0^4\, \beta^2 + \left(1 - c0^2\, \beta^2\right)\left(3 - 3\, c0^2\left(1 - \beta^2\right)\right)$

Out[931]= $8 - 4\, c0^2 - c0^4\left(\beta^2 + 3\,\beta^4\right)$

Out[932]= $$\frac{e^2\,\beta^3\left(8 - 4\, c0^2 - c0^4\left(\beta^2 + 3\,\beta^4\right)\right)}{16\, R\, T\left(1 - c0^2\,\beta^2\right)^{7/2}}$$

The Mathematica calculations above prove that EQ (11.293) implies EQ (11.294) after doing the integral over ϕ from 0 to 2π.

Notes:

(1) I did the integral from 0 to π and then multiplied by 2.

(2) the variable c0 is equal to cos(θ).

Part (b)

In[1080]:= `eq294`

Out[1080]= $$\frac{e^2\,\beta^3\left(8 - 4\, c0^2 - c0^4\left(\beta^2 + 3\,\beta^4\right)\right)}{16\, R\, T\left(1 - c0^2\,\beta^2\right)^{7/2}}$$

```
(* limit of the denominator *)
ξ1 = 1 - c0^2 * β^2
ξ1 = ξ1 /. {c0 → 1 - θ^2 / 2, β → Sqrt[1 - γm2]};
ξ1 = ξ1 // Expand // Simplify;
ξ1 = Series[ξ1, {θ, 0, 2}] // Normal // Expand;
ξ1 = ξ1 /. { γm2 * θ^2 → 0};
ξ1 = Normal[ξ1];
den = 16 * R * T * ξ1^(7 / 2)
```

Out[1058]= $1 - c0^2\,\beta^2$

Out[1064]= $16\, R\, T\left(\gamma m2 + \theta^2\right)^{7/2}$

```
(* limit of the numerator *)
ξ2 = 8 - 4 * c0^2 - c0^4 * (β^2 + 3 * β^4)
ξ2 = ξ2 /. {c0 → 1 - θ^2 / 2, β → Sqrt[1 - γm2]}
ξ2 = ξ2 /. {θ^4 → 0};
ξ2 = ξ2  /. {θ^2 → δ * th^2, γm2 →  δ * gm2};
ξ2 = ξ2 // Expand
Series[ξ2, {δ, 0, 2}]
ξ2 = 7 * γm2 + 12 * θ^2
num = e^2 * ξ2
```

Out[1092]= $8 - 4\ \text{c0}^2 - \text{c0}^4\ \left(\beta^2 + 3\ \beta^4\right)$

Out[1093]= $8 - 4\ \left(1 - \frac{\theta^2}{2}\right)^2 - \left(1 + 3\ (1 - \gamma\text{m2})^2 - \gamma\text{m2}\right)\ \left(1 - \frac{\theta^2}{2}\right)^4$

Out[1096]= $7\ \text{gm2}\ \delta + 12\ \text{th}^2\ \delta - 3\ \text{gm2}^2\ \delta^2 - 14\ \text{gm2}\ \text{th}^2\ \delta^2 - 7\ \text{th}^4\ \delta^2 + 6\ \text{gm2}^2\ \text{th}^2\ \delta^3 + \frac{21}{2}\ \text{gm2}\ \text{th}^4\ \delta^3 + 2\ \text{th}^6\ \delta^3 - \frac{9}{2}\ \text{gm2}^2\ \text{th}^4\ \delta^4 - \frac{7}{2}\ \text{gm2}\ \text{th}^6\ \delta^4 - \frac{\text{th}^8\ \delta^4}{4} + \frac{3}{2}\ \text{gm2}^2\ \text{th}^6\ \delta^5 + \frac{7}{16}\ \text{gm2}\ \text{th}^8\ \delta^5 - \frac{3}{16}\ \text{gm2}^2\ \text{th}^8\ \delta^6$

Out[1097]= $\left(7\ \text{gm2} + 12\ \text{th}^2\right)\ \delta + \left(-3\ \text{gm2}^2 - 14\ \text{gm2}\ \text{th}^2 - 7\ \text{th}^4\right)\ \delta^2 + O[\delta]^3$

Out[1098]= $7\ \gamma\text{m2} + 12\ \theta^2$

Out[1099]= $\text{e}^2\ \left(7\ \gamma\text{m2} + 12\ \theta^2\right)$

In[1111]:=
```
(* limit of the power *)
myresult = num / den
```

Out[1111]= $\frac{\text{e}^2\ \left(7\ \gamma\text{m2} + 12\ \theta^2\right)}{16\ \text{R}\ \text{T}\ \left(\gamma\text{m2} + \theta^2\right)^{7/2}}$

In[1121]:=
```
(* comparison to the given expression *)
(* the given expression is not quite right; *)
(* we need to replace cT by R *)
comparison = (e^2 * ω0) / (2 π * R) *
  7 / 16 * (γm2 + θ^2) ^ (-7 / 2) * (γm2 + θ^2 + 5 / 7 * θ^2)
comparison = comparison /. {ω0 → 2 Pi / T}
comparison // Expand // Simplify
```

Out[1121]= $\frac{7\, e^2 \left(\gamma m2 + \frac{12\, \theta^2}{7}\right) \omega 0}{32\, \pi\, R \left(\gamma m2 + \theta^2\right)^{7/2}}$

Out[1122]= $\frac{7\, e^2 \left(\gamma m2 + \frac{12\, \theta^2}{7}\right)}{16\, R\, T \left(\gamma m2 + \theta^2\right)^{7/2}}$

Out[1123]= $\frac{e^2 \left(7\, \gamma m2 + 12\, \theta^2\right)}{16\, R\, T \left(\gamma m2 + \theta^2\right)^{7/2}}$

Thus the limit of the result in (a) agrees with the corrected expression in (b).

Problem 9-6 = Exercise 11.9.3

In[1134]:= ex93

Out[1134]=

Exercise 11.9.3. Integrate expression (11.294) for $dP/d\Omega$ over angles either by a symbolic integration package or explicitly. If explicitly, you may use the following integral formulas from Gradshteyn and Rhyzik, *op. cit.*, 2.271(6), 2.272(6), and 2.273(7):

$$\int \frac{dx}{u^{2n+1}} = \frac{1}{a^n} \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} \binom{n-1}{k} \frac{c^k x^{2k+1}}{u^{2k+1}},$$

$$\int \frac{x^2 dx}{u^{2n+1}} = \frac{1}{a^{n-1}} \sum_{k=0}^{n-2} \frac{(-1)^k}{2k+3} \binom{n-2}{k} \frac{c^k x^{2k+3}}{u^{2k+3}},$$

$$\int \frac{x^4 dx}{u^{2n+1}} = \frac{1}{a^{n-2}} \sum_{k=0}^{n-3} \frac{(-1)^k}{2k+5} \binom{n-3}{k} \frac{c^k x^{2k+5}}{u^{2k+5}},$$

where

$$u \equiv \sqrt{a + cx^2},$$

$$\binom{n}{k} \equiv \frac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdots k}, \quad \binom{n}{0} \equiv 1.$$

Start with EQ (11.294).

In this equation, ***θ* is not the polar angle**—i.e., the angle with the z axis.
See EQ (11.290), which implies that $\theta = \pi/2 - \alpha$, where α is the usual polar angle.
Recall that α goes from 0 to π; therefore θ goes from $-\pi/2$ to $+\pi/2$.

To do the integral, $d\Omega = 2\pi \sin\alpha \; d\alpha$;
and we must integrate over α from 0 to π.
Or, let $u = \cos\alpha$; then $\sin\alpha \; d\alpha = du$, and u goes from -1 to 1.

Also, in the integrand, replace $\cos^2\theta = \cos^2(\pi/2-\alpha) = \sin^2\alpha = 1 - u^2$.

In[1272]:=
```
Remove[f1, f2, g1, g2]
f1 = 3 - 3 * c0^2 * (1 - β^2) /. {c0^2 → 1 - u^2};
f2 = 5 - c0^2 - 4 β^2 * c0^4 /. {c0^2 → 1 - u^2, c0^4 → (1 - u^2)^2};
g1 = (1 - β^2 * c0^2)^(5 / 2) /. {c0^2 → 1 - u^2};
g2 = (1 - β^2 * c0^2)^(7 / 2) /. {c0^2 → 1 - u^2};
fI = Integrate[f1 / g1 + f2 / g2, u];
fIU = fI /. u → 1;
fIL = fI /. u → -1;
Q = 2 π (fIU - fIL);
Q = Q // Expand // Simplify
```

Out[1281]=
$$\frac{64\,\pi}{3\,\left(-1+\beta^2\right)^2}$$

The result is

$$P = \frac{e^2\,\beta^3}{16\,R\,T}\;\frac{64\,\pi}{3\,(\,1-\beta^2\,)^2} = \frac{4\,\pi\,e^2}{3\,\mathrm{RT}}\;\frac{\beta^3}{(1-\beta^2)^2}$$

which does agree with EQ (11.171).