Assignment 10
Solutions for J Morrison

Assignment 10

Problem 1 = Exercise 13.7.1 (8 points)

(a) In the rest frame of the particle (frame $\mathcal{F}'$)

dE' = energy radiated in time dt'; also $d\vec{p}\,' = 0$ and $d\vec{x}\,' = 0$

In the lab frame $dE = \gamma\, dE'$ and $dt = \gamma\, dt'$.

Thus $$\frac{dE}{dt} = \frac{dE'}{dt'} \quad \text{(invariant)}$$ 2 points

(b) $$\alpha^\mu = \text{4-vector acceleration} = \frac{1}{m}\frac{dp^\mu}{d\tau} = \frac{1}{m}\gamma\frac{dp^\mu}{dt}$$

- $$\alpha^0 = \frac{\gamma}{m}\frac{d}{dt}(\gamma m c) = \gamma c \frac{d\gamma}{dt}$$

$$\boxed{\begin{aligned}\frac{d\gamma}{dt} &= \frac{d}{dt}\frac{1}{\sqrt{1-v^2/c^2}} \\ &= \frac{-\frac{1}{2}(-2\vec{v}\cdot\dot{\vec{v}}/c^2)}{(1-v^2/c^2)^{3/2}} \\ &= \gamma^3\,\vec{\beta}\cdot\frac{\dot{\vec{v}}}{c}\end{aligned}}$$

$$\alpha^0 = \gamma^4\,\vec{\beta}\cdot\dot{\vec{v}}$$ 1 pt

and

- $$\vec{\alpha} = \frac{\gamma}{m}\frac{d}{dt}(\gamma m\vec{v}) = \gamma\left(\gamma\frac{d\vec{v}}{dt} + \vec{v}\frac{d\gamma}{dt}\right)$$

$$= \gamma^2\dot{\vec{v}} + \gamma\vec{v}\,\gamma^3\vec{\beta}\cdot\frac{\dot{\vec{v}}}{c} = \gamma^2\left[\dot{\vec{v}} + \gamma^2\underbrace{\vec{\beta}(\vec{\beta}\cdot\dot{\vec{v}})}_{\vec{\beta}\times(\vec{\beta}\times\dot{\vec{v}}) + \beta^2\dot{\vec{v}}}\right]$$

$$= \gamma^2\left[\underbrace{(1+\gamma^2\beta^2)}_{\frac{1-\beta^2+\beta^2}{1-\beta^2} = \gamma^2}\dot{\vec{v}} + \gamma^2\,\vec{\beta}\times(\vec{\beta}\times\dot{\vec{v}})\right]$$

$$= \gamma^4\left[\dot{\vec{v}} + \vec{\beta}\times(\vec{\beta}\times\dot{\vec{v}})\right]$$ 1 pt

- After a bit of algebra, 2 pts

$$\alpha^\mu\alpha_\mu = (\alpha^0)^2 - (\vec{\alpha})^2 = -\gamma^6 c^2\left[\dot{\vec{\beta}}^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2\right]$$

The algebra —

$$\alpha^\mu \alpha_\mu = (\alpha^0)^2 - (\vec{\alpha})^2$$

$$= (\gamma^4 \vec{\beta}\cdot\dot{\vec{v}})^2 - (\gamma^4 [\dot{\vec{v}} + \vec{\beta}\times(\vec{\beta}\times\dot{\vec{v}})])^2$$

$$= \gamma^8 \{ (\vec{\beta}\cdot\dot{\vec{v}})^2 - [\dot{\vec{v}} + \vec{\beta}\times(\vec{\beta}\times\dot{\vec{v}})]^2 \}$$

$$= \gamma^8 \{ (\vec{\beta}\cdot\vec{a})^2 - [\vec{a} + \vec{\beta}(\vec{\beta}\cdot\vec{a}) - \beta^2\vec{a}]^2 \}$$

$$= \gamma^8 \{ (\vec{\beta}\cdot\vec{a})^2 - [(1-\beta^2)\vec{a} + \vec{\beta}(\vec{\beta}\cdot\vec{a})]^2 \}$$

$$= \gamma^8 \{ (\vec{\beta}\cdot\vec{a})^2 - (1-\beta^2)^2 a^2 - \beta^2(\vec{\beta}\cdot\vec{a})^2 - 2(1-\beta^2)(\vec{\beta}\cdot\vec{a})^2 \}$$

$$= \gamma^8 \{ -(1-\beta^2)^2 a^2 + [1-\beta^2-2(1-\beta^2)](\vec{\beta}\cdot\vec{a})^2 \}$$

$$= \gamma^6 \{ -(1-\beta^2)a^2 - (\vec{\beta}\cdot\vec{a})^2 \}$$

$$= -\gamma^6 c^2 \{ (1-\beta^2)\dot{\vec{\beta}}^2 + (\vec{\beta}\cdot\dot{\vec{\beta}})^2 \}$$

$$= -\gamma^6 c^2 \{ \dot{\vec{\beta}}^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2 \}$$ as claimed

(c) <u>Larmor formula</u>

In the rest frame of the particle,

$$P' = \frac{dE'}{dt'} = \frac{2}{3}\frac{e^2}{c^3}(\vec{a}' \cdot \vec{a}') \qquad (11.99)$$

This must be Lorentz invariant, so it must be proportional to $a^\mu a_\mu$, which is $-\vec{a}' \cdot \vec{a}'$ in the rest frame. Thus

$$P = \frac{2}{3}\frac{e^2}{c^3}(-a^\mu a_\mu) \qquad (\text{eq } 100)$$

$$= \frac{2}{3}\frac{e^2}{c^3}\gamma^6 \left\{ \dot{\vec{v}}^2 - (\vec{\beta} \times \dot{\vec{v}})^2 \right\} \qquad (\text{eq } 11.100)$$

<u>2 pts</u>

Problem 2 = Exercise 13.7.2 $\boxed{\text{6 points}}$

(a) $\frac{d\vec{p}}{dt} = e\vec{E} = eE_0\,\hat{e}_x$ where $\vec{p} = \gamma m \vec{v}$
$= \gamma m \frac{d\vec{x}}{dt}$

$$\frac{d}{dt}(\gamma m v_x) = eE_0$$

$$\therefore \gamma v_x = \frac{eE_0}{m}t = \frac{v}{\sqrt{1-v^2/c^2}}$$

$$v = \frac{eE_0 t/m}{\sqrt{1+(eE_0 t/mc)^2}}$$

$$x = \int_0^t v\,dt = \frac{mc^2}{eE_0}\left[\sqrt{1+\left(\frac{eE_0 t}{mc}\right)^2} - 1\right]$$

or, $$\left(x + \frac{mc^2}{eE_0}\right)^2 = c^2t^2 + \left(\frac{mc^2}{eE_0}\right)^2$$

(b) Retarded time $t_R = t - \frac{x(t_R)}{c}$. (i.e. observer is at the origin)

$$ct_R = ct - x(t_R) = ct + \left\{\frac{mc^2}{eE_0}\right\}\left[1-\sqrt{1+\left(\frac{eE_0 t_R}{mc}\right)^2}\right]$$

Solve for t_R

$$\left(\frac{eE_0}{mc^2}\right)^2(ct_R - ct)^2 = 1 + \left(\frac{eE_0 t_R}{mc}\right)^2$$ 2 points

After a bit of algebra

$$t_R = t\left[\frac{t^2+\zeta t - 2\zeta^2}{2(t^2-\zeta^2)}\right]$$ where $\zeta = \frac{mc}{eE_0}$ 2 pts

For $t \ll \zeta$, $t_R \simeq t$; For $t \gg \zeta$, $t_R \simeq t/2$ 2 pts

The algebra —

$$c t_R = ct - x(t_R) = ct - \left[-\frac{mc^2}{eE_0} + \sqrt{c^2 t_R^2 + \left(\frac{mc^2}{cE_0}\right)^2} \right]$$

Let $\zeta = \frac{mc}{eE_0}$; then

$$t_R = t - \left[-\zeta + \sqrt{t_R^2 + \zeta^2} \right]$$

$$= t + \zeta - \sqrt{t_R^2 + \zeta^2}$$

$$\sqrt{t_R^2 + \zeta^2} = t + \zeta - t_R$$

$$\underset{\uparrow \text{cancel}}{t_R^2} + \zeta^2 = (t+\zeta)^2 + \underset{\uparrow \text{cancel}}{t_R^2} - 2 t_R (t+\zeta)$$

$$2 t_R (t+\zeta) = (t+\zeta)^2 - \zeta^2 = t^2 + 2t\zeta$$

$$t_R = \frac{t(t+2\zeta)}{2(t+\zeta)} = \frac{t}{2} \frac{(t+2\zeta)(t-\zeta)}{(t+\zeta)(t-\zeta)}$$

$$= \frac{t}{2} \frac{t^2 + \zeta t - 2\zeta^2}{t^2 - \zeta^2} \quad \text{where } \zeta = \frac{mc}{eE_0}$$

- For $t \ll \zeta$, $t_R \approx t$
- For $t \gg \zeta$, $t_R \approx \frac{t}{2}$

Problem 3 = Exercise 13.7.3 $\boxed{8 \text{ points}}$

(a) $$\frac{dp^\mu}{d\tau} = \frac{e}{c} F^\mu{}_\nu u^\nu$$

where $p^\mu = m u^\mu$ and $d\tau$ = proper time ; $u^\nu = \frac{dx^\nu}{d\tau}$

(a) • Spatial components ($\mu = 1, 2, 3$)

$$\frac{dp^k}{dt} = \frac{dp^k}{d\tau}\frac{d\tau}{dt} = \frac{e}{c} F^k{}_\nu u^\nu \left(\frac{d\tau}{dt}\right) = \frac{e}{c} F^k{}_\nu \frac{dx^\nu}{dt}$$

$$= \frac{e}{c}\Big[\underbrace{F^k{}_0}_{E^k} \underbrace{\frac{dx^0}{dt}}_{c} + \underbrace{F^k{}_i}_{\epsilon_{kij} B^j} \underbrace{\frac{dx^i}{dt}}_{v^i}\Big]$$

$$= e\left[E^k + \left(\frac{\vec{v}}{c} \times \vec{B}\right)^k \right]$$

2 points

$$\boxed{\vec{F} = \frac{d\vec{p}}{dt} = e\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)}$$

• Time component ($\mu = 0$)

$$\frac{dp^0}{dt} = \frac{e}{c} F^0{}_\nu \frac{dx^\nu}{dt} = \frac{e}{c} E^i \frac{dx^i}{dt} \quad \text{since } F^{00} = 0$$

$$\frac{1}{c}\frac{dE}{dt} = \frac{e}{c}\, \vec{E} \cdot \vec{v}$$

2 pts

$$\boxed{\frac{dE}{dt} = e\, \vec{v} \cdot \vec{E}}$$

(b) $$P = -\frac{2}{3}\frac{e^2}{m^2c^3}\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau}$$

$$= \frac{-2}{3}\frac{e^2}{m^2c^3}\left\{\frac{1}{c^2}\left(\frac{dE}{dt}\right)^2-\left(\frac{d\vec{p}}{dt}\right)^2\right\}\left(\frac{dt}{d\tau}\right)^2$$

$$= -\frac{2}{3}\frac{e^2}{m^2c^3}\left\{\frac{1}{c^2}e^2v^2E_{\parallel}^2 - e^2\underbrace{\left(\vec{E}+\frac{\vec{v}}{c}\times\vec{B}_\perp\right)^2}\right\}\gamma^2$$

$$= E_{\parallel}^2 + \left(\vec{E}_\perp+\frac{\vec{v}}{c}\times\vec{B}_\perp\right)^2$$

$$= E_{\parallel}^2 + \vec{E}_\perp^2 + \left(\frac{\vec{v}}{c}\times\vec{B}_\perp\right)^2$$

$$= E_{\parallel}^2 + \vec{E}_\perp^2 + \frac{v^2}{c^2}\vec{B}_\perp^2$$

$$= -\frac{2}{3}\frac{e^4}{m^2c^3}\gamma^2\left\{\beta^2E_{\parallel}^2 - \left(E_{\parallel}^2+E_\perp^2+\frac{v^2}{c^2}B_\perp^2\right)\right\}$$

$$= -\frac{2}{3}\frac{e^4}{m^2c^3}\gamma^2\left\{-(1-\beta^2)E_{\parallel}^2 - \left(E_\perp^2+\frac{v^2}{c^2}\vec{B}_\perp^2\right)\right\}$$

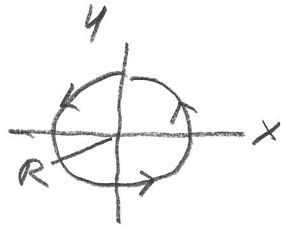
$$= \frac{2}{3}\frac{e^4}{m^2c^3}\left\{-E_{\parallel}^2 - \gamma^2\left(E_\perp^2+\frac{v^2}{c^2}\vec{B}_\perp^2\right)\right\}$$

- If $\vec{B}=0$ and $\vec{E}_\perp=0$: $\quad P_E = \frac{-2}{3}\frac{e^4}{m^2c^3}\vec{E}^2$ 2 pts
- If $\vec{E}=0$ and $\vec{B}\perp\vec{v}$:

$$P_B = -\frac{2}{3}\frac{e^4}{m^2c^3}\gamma^2\beta^2\vec{B}^2$$ 2 pts

Problem 10-4 = Exercise 13.7.4 $\boxed{4 \text{ points}}$

(a)



The Lorentz force law is

$$\frac{d\vec{p}}{dt} = \frac{e}{c}\vec{v} \times \vec{B} \quad , \quad \vec{B} = B_z \hat{e}_z$$

For circular motion,

$$\vec{x} = R\left[\hat{e}_x \cos\omega t + \hat{e}_y \sin\omega t\right]$$

$$\vec{v} = R\omega\left[-\hat{e}_x \sin\omega t + \hat{e}_y \cos\omega t\right]$$

$$\frac{d\vec{v}}{dt} = -R\omega^2\left[\hat{e}_x \cos\omega t + \hat{e}_y \sin\omega t\right]$$

Now, using relativity, $\vec{p} = \gamma m \vec{v}$ where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$.

$$\frac{d\vec{p}}{dt} = m\gamma\frac{d\vec{v}}{dt} + m\vec{v}\frac{d\gamma}{dt} = m\gamma\frac{d\vec{v}}{dt} \text{ because } \frac{d\gamma}{dt} = 0.$$

$$-m\gamma R\omega^2\left(\hat{e}_x\cos\omega t + \hat{e}_y \sin\omega t\right) = \frac{e}{c}(R\omega)\left(-\hat{e}_x \sin\omega t + \hat{e}_y\cos\omega t\right)\times B_z\hat{e}_z$$

$$= \frac{e}{c}R\omega B_z\left(\hat{e}_y \sin\omega t + \hat{e}_x \cos\omega t\right)$$

$$-m\gamma\omega = \frac{e}{c}B_z \quad \text{and} \quad \vec{p} = \gamma m\vec{v} \Rightarrow p = \gamma m R\omega$$

$$\frac{e}{c}B_z = -\frac{p}{R}$$

$$B_z = \frac{-cp}{eR} \quad \text{so} \quad |\vec{B}| = \frac{cp}{eR} \quad \text{or} \quad \boxed{R = \frac{cp}{eB}}$$ 2 points

(need to have a relativistic calculation)

(b) Now, $P_B = -\frac{2}{3}\frac{e^4}{m^2c^3}\gamma^2\beta^2 B^2$ from Ex. 13.7.3.

Calculate the energy loss per cycle =

$$E_{cycle} = (-P_B)T = (-P_B)\left(\frac{2\pi}{\omega}\right)$$

$$= \left(\frac{2\pi}{\omega}\right)\frac{2}{3}\frac{e^4}{m^2c^3}\gamma^2\beta^2\left(\frac{cp}{eR}\right)^2 \quad \text{where } p = \gamma m v = \gamma m R\omega$$

$$= \frac{4\pi}{3\omega}\frac{e^2}{m^2c^3}\gamma^2\beta^2\left(\frac{c\gamma m R\omega}{R}\right)^2$$

$$= \frac{4\pi e^2}{3m^2c^3}\gamma^2\beta^2(c\gamma m)^2\,\omega \quad \text{where } \omega = \frac{v}{R} = \frac{c\beta}{R}$$

$$= \frac{4\pi e^2}{3}\frac{\gamma^4\beta^2(mc)^2}{m^2c^3}\frac{c\beta}{R} \quad \text{where } \gamma = \frac{E}{mc^2}$$

$$= \frac{4\pi e^2}{3R}\left(\frac{E}{mc^2}\right)^4\beta^3$$

2 points

Problem 10-5 : Exercise 13.7.5 (4 points)

Energy loss per cycle $= \Delta = \frac{4\pi e^2}{3R}\left(\frac{E_e}{mc^2}\right)^4 \beta^3$

<u>Part 1</u> $\frac{dE}{dx} = \frac{\Delta}{2\pi R}$

For $E_e = 100$ GeV,

$$\frac{dE}{dx} = 1.22\times10^{-2} \text{ erg/cm} = 761 \text{ eV/cm}$$ (2 points)

<u>Part 2</u> For $\frac{dE}{dx} = 50 \frac{\text{MeV}}{\text{m}} = 5\times10^5$ eV/cm

the electron energy would be 506 GeV (= 0.81 erg)

(2 points)