Assignment #11
J Morrison to grade #1, 2, 3

Homework Assignment #11 due Friday November 9

Pr 11-1 : Exercise 12.1.1

Pr 11-2 : Exercise 12.3.1

Pr 11-3 : Exercise 12.3.2

Pr 11-4 : Exercise 12.3.3

1.) 4

2.) 14

3.) 12

Total: 30

Pr 11-1 : Exercise 12.1.1

In[195]:= **E11**

Exercise 12.1.1. Use the model of Section 9.5 and the harmonic results in (11.112) to discuss the scattering from free or bound electrons in a medium in electric dipole approximation. Using a linearly polarized incoming wave and integrating over angles, show that one obtains the cross section

$$\sigma(\omega) \equiv \int d\Omega \left(\frac{dP}{d\Omega}\right)_{\text{avg}} / |\vec{S}_0| = \frac{8\pi}{3} r_e^2 f(\omega),$$

where $r_e \equiv e^2/(mc^2)$ is called the *classical electron radius*

Out[195]= (encountered again in Chapter 14), and where

$$f(\omega) \equiv \frac{\omega^4}{[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]},$$

which gives a resonance in the cross section at $\omega \approx \omega_0$. The low and high energy forms of this cross section are

$$\omega \ll \omega_0, \frac{\omega_0^2}{\gamma}: \quad \sigma(\omega) \approx \frac{8\pi}{3} r_e^2 \left(\frac{\omega}{\omega_0}\right)^2,$$
$$\omega \gg \omega_0, \gamma: \quad \sigma(\omega) \approx \frac{8\pi}{3} r_e^2.$$

Exercise 12.1.1 (4 points)

Start with the electric dipole radiation formula, Eq. (11.112)

$$\frac{dP}{d\Omega} = \frac{ck^4}{8\pi}|\hat{n}\times\vec{p}|^2 \quad ; \quad k = \frac{\omega}{c}$$

From Chapter 9:

Eq. (9.130) $\vec{x}(t) = \frac{e}{m}\,\mathrm{Re}\left[\frac{\vec{\mathcal{E}}(\omega)e^{-i\omega t}}{\omega_0^2-\omega^2-i\omega\gamma}\right]$

and the dipole moment is $\vec{p} = e\vec{x}$

$$\Rightarrow \quad \vec{p}(\omega) = \frac{e^2}{m}\frac{\vec{\mathcal{E}}}{\omega_0^2-\omega^2-i\omega\gamma}$$

For an incident plane wave, $\vec{\mathcal{E}} = \mathcal{E}_0\hat{\epsilon}_0 e^{i\vec{k}_0\cdot\vec{x}}$ $\quad k = \frac{\omega}{c}$

$$\therefore \frac{dP}{d\Omega} = \frac{ck^4}{8\pi}\frac{e^4}{m^2}\frac{\mathcal{E}_0^2(\hat{n}\times\hat{\epsilon}_0)^2}{|\omega_0^2-\omega^2-i\omega\gamma|^2} = \underbrace{\frac{c\mathcal{E}_0^2}{8\pi c^4}\frac{e^4}{m^2}(\hat{n}\times\hat{\epsilon}_0)^2 f(\omega)}_{\text{2 points}}$$

$$\text{where } f(\omega) = \frac{\omega^4}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}$$

The total cross section is

$r_e = \frac{e^2}{mc^2}$ $\quad r_e^2 = \frac{e^4}{m^2c^4}$

$$\sigma = \frac{1}{|\vec{S}_0|}\int d\Omega\frac{dP}{d\Omega} \quad \text{where } |\vec{S}_0| = \frac{c}{8\pi}\mathcal{E}_0^2$$

$$\sigma = \frac{8\pi}{c\mathcal{E}_0^2}\frac{c\mathcal{E}_0^2}{8\pi}\left(\frac{e^2}{mc^2}\right)^2 f(\omega)\int d\Omega\,(\hat{n}\times\hat{\epsilon}_0)^2$$

$$= r_e^2 f(\omega)\int d\Omega\left[\hat{n}^2\hat{\epsilon}_0^2 - (\hat{n}\cdot\hat{\epsilon}_0)^2\right]$$

The angular integral $= \int d\Omega(1-(\hat{n}\cdot\hat{\epsilon}_0)^2) = \int d\Omega(1-\cos^2\theta)$

$$= 2\pi\int_0^\pi \sin\theta\, d\theta\,(1-\cos^2\theta) = 2\pi\int_{-1}^1 du\,(1-u^2) = 2\pi(2-2/3)$$

$$= 8\pi/3$$

$$\sigma = \underbrace{\frac{8\pi}{3}r_e^2 f(\omega)}_{\text{2 points}}, \quad \text{where } f(\omega) = \frac{\omega^4}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}$$

Pr 11-2 : Exercise 12.3.1

In[198]:= E31

Exercise 12.3.1. The induced charges and currents on the surface of the spherical conductor are given by (12.58) and (12.59), using standard harmonic incident plane wave fields $\vec{E}_{\text{inc}}$ and $\vec{B}_{\text{inc}}$ with initial wave number $\vec{k}_0$ and polarization vector $\hat{\epsilon}_0$.

(a) Make sure that (12.58) and (12.59) form a conserved combination by showing the continuity equation is satisfied for the given incident fields.

Out[198]=

(b) Use an adapted Equation (12.34) for the scattering amplitude to calculate the polarized scattering amplitude induced by the surface current $\vec{K}(\vec{r}\,')$. Show that it gives the shadow (forward) scattering amplitude, Equation (12.64). Thus, we can regain the entire scattering amplitude by integrations over just the illuminated portion of the sphere!

Exercise 12.3.1 (4+10 = 14 points total)

12.58 $\quad \vec{K} = \frac{c}{4\pi}\, \hat{n}' \times (\vec{B}_{inc} + \vec{B}_{sc})$

12.59 $\quad \sigma = \frac{1}{4\pi}\, \hat{n}' \cdot (\vec{E}_{inc} + \vec{E}_{sc})$

- On the illuminated side, $\vec{K} = \frac{2c}{4\pi}\, \hat{n}' \times \vec{B}_{inc}$ and $\sigma = \frac{2}{4\pi}\, \hat{n}' \cdot \vec{E}_{inc}$
- On the shadow side, $\vec{K} = 0$ and $\sigma = 0$.

(a) The continuity equation in a volume is $\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$. (4 points)

On a surface, $\nabla^{(2)} \cdot \vec{K} = -\frac{\partial \sigma}{\partial t} = i\omega\sigma$

Now verify this on the illuminated part of S, using $\vec{E}_{inc} = \hat{\epsilon}_0 E_0 e^{i(\vec{k}_0 \cdot \vec{x} - \omega t)}$

$$\vec{B}_{inc} = \hat{k}_0 \times \vec{E}_{inc} = \hat{k}_0 \times \hat{\epsilon}_0\, E_0\, e^{i(\vec{k}_0 \cdot \vec{x} - \omega t)}$$

$$\nabla'^{(2)} \cdot \vec{K}(\vec{x}') = \frac{2c}{4\pi}\, \underbrace{\nabla'}_{i\vec{k}_0} \cdot \left[\hat{n}' \times (\hat{k}_0 \times \hat{\epsilon}_0)\right] E_0\, e^{i(\vec{k}_0 \cdot \vec{x}' - \omega t)}$$

$$= \frac{2cE_0}{4\pi}\, i\vec{k}_0 \cdot \left[\hat{k}_0 (\hat{n}' \cdot \hat{\epsilon}_0) - \hat{\epsilon}_0 (\hat{n}' \cdot \hat{k}_0)\right] e^{i\Phi}$$

$$= \underbrace{\frac{2cE_0}{4\pi} \left(\frac{i\omega}{c}\right) \left[\hat{n}' \cdot \hat{\epsilon}_0\right] e^{i\Phi}} \quad \leftarrow \text{2 points}$$

and compare

$$i\omega\sigma = i\omega \frac{2}{4\pi}\, \hat{n}' \cdot \hat{\epsilon}_0\, E_0\, e^{i\Phi} = \frac{2cE_0}{4\pi}\left(\frac{i\omega}{c}\right) \hat{n}' \cdot \hat{\epsilon}_0\, e^{i\Phi}$$

$$= \nabla'^{(2)} \cdot \vec{K}(\vec{x}') \quad ✓ \quad \leftarrow \text{2 points}$$

$$= -\frac{\partial \sigma}{\partial t}$$

12.3.1 part (b)

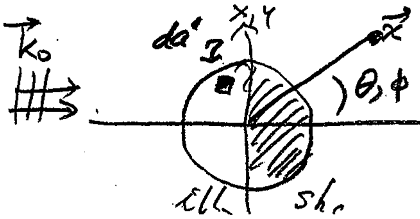
Use Eq. (12.34) but replace the volume integral by a surface integral

$$A = \vec{\epsilon}_f \cdot \vec{f} = \frac{2k}{cE_0}\,\hat{e}_f \cdot \int_{S_{ill}} da'\,\vec{K}(\vec{x}')\,e^{-i\vec{k}\cdot\vec{x}'}$$

where

$\vec{K}(\vec{x}') = \frac{c}{4\pi}\,\hat{n}' \times (2\vec{B}_{inc})$ on the illuminated side only

Also, $\vec{k} = k\hat{n}$



$\rightarrow \vec{k}_0 = \frac{\omega}{c}\hat{e}_z$

$\vec{k}_0 = k\hat{e}_z$

$\vec{k} = k\{\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\}$

$\hat{n}' = \{\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta'\}$

$\vec{x}' = a\hat{n}'$

$$\int_{ill} da' = a^2 \int_{\pi/2}^{\pi} \sin\theta'\,d\theta' \int_0^{2\pi} d\phi' \;(\ldots)$$

$$\vec{K}(\vec{x}') = \frac{2c}{4\pi} E_0\, \hat{n}' \times (\hat{k}_0 \times \hat{\epsilon}_0)\, e^{i\vec{k}_0\cdot\vec{x}'}$$

2 points

$$\underbrace{A = \frac{ik}{c E_0}\,\frac{2c}{4\pi}E_0 \int_{ill} a^2\,d\Omega'\;\hat{\epsilon}_f \cdot \left[\hat{n}' \times (\hat{k}_0 \times \hat{\epsilon}_0)\right] e^{-i(\vec{k}-\vec{k}_0)\cdot\vec{x}'}}$$

The factor $e^{-i(\vec{k}-\vec{k}_0)\cdot\vec{x}'}$ is rapidly oscillating unless $\vec{k} \approx \vec{k}_0$

We can make 2 approximations:

(i) $\hat{\epsilon}_f \cdot [\hat{n}' \times (\vec{k}_0 \times \hat{\epsilon}_0)] = \hat{\epsilon}_f \cdot [\vec{k}_0\,\hat{n}'\cdot\hat{\epsilon}_0 - \hat{\epsilon}_0\,\hat{n}'\cdot\vec{k}_0]$ (neglect)

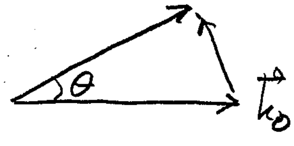
$\approx -\hat{\epsilon}_f \cdot \hat{\epsilon}_0\;\hat{n}'\cdot\vec{k}_0$ because $\hat{\epsilon}_f \cdot \vec{k}_0 \approx \hat{\epsilon}_f \cdot \vec{k} = 0$.

(ii) $e^{-i(\vec{k}-\vec{k}_0)\cdot\vec{x}'} = e^{-i(k_z - k_{0z})z'}\; e^{-i(\vec{k}_\perp - \vec{k}_{0\perp})\cdot\vec{x}'_\perp}$

$= e^{-ik\underbrace{(\cos\theta - 1)}_{}z'}\; e^{-i(\vec{k}_\perp - \vec{k}_{0\perp})\cdot\vec{x}'_\perp}$

$O(\theta^2)$ is negligible because $\vec{k} \approx \vec{k}_0$.

$\approx e^{-i(\vec{k}-\vec{k}_0)_\perp\cdot\vec{x}'_\perp}$



$\vec{q} = \vec{k} - \vec{k}_0 = k\{\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta - 1\}$ (neglect)

$e^{-i\vec{q}_\perp\cdot\vec{x}'_\perp} = e^{-ika[\sin\theta\,\sin\theta'\,\cos(\phi-\phi')]}$

Exercise 12.3.1 part (b) continues

$$A = \frac{-ik}{c\epsilon_0}\frac{2c}{4\pi}E_0 a^2 \overbrace{\left(\vec{\epsilon}_f\cdot\vec{\epsilon}_0\right)}^{2\text{ points}} \underbrace{\int_{\text{ill.}} d\Omega'\, \overbrace{\hat{n}\cdot\hat{n}_0\, e^{-i\vec{q}_\perp\cdot\vec{x}'}}^{2\text{ points}}}$$

$$= \frac{-2ika^2}{4\pi}\int_{\pi/2}^{\pi}\sin\theta\, d\theta' \int_0^{2\pi} d\phi' \cos\theta'\, e^{-ika\sin\theta\sin\theta'\cos(\phi-\phi')} \quad \vec{\epsilon}_f\cdot\vec{\epsilon}_0$$

Bessel function

$$\int_0^{2\pi} d\varphi\, e^{i\tau\cos\varphi} = 2\pi J_0(\tau)$$

$$= -ika^2\int_{\pi/2}^{\pi} d\theta'\,\sin\theta'\cos\theta'\,\underline{J_0(ka\sin\theta\sin\theta')} \quad \vec{\epsilon}_f\cdot\vec{\epsilon}_0$$

2 points for Bessel fun. J_0

Let $u = \sin\theta'$

$du = \cos\theta' d\theta'$; $u \in (0,1)$

$$= ika^2\int_0^1 du\, u\, J_0(ka\sin\theta\, u) \quad \vec{\epsilon}_f\cdot\vec{\epsilon}_0$$

$$= ika^2\int_0^{ka\sin\theta} dx\, x\, J_0(x)\,\frac{1}{(ka\sin\theta)^2} \quad \vec{\epsilon}_f\cdot\vec{\epsilon}_0$$

$$J_0(x) = \frac{1}{x}\frac{d}{dx}\left(x J_1(x)\right)$$ derivative of Bessel function

$$A = ika^2\,\frac{ka\sin\theta\, J_1(ka\sin\theta)}{(ka\sin\theta)^2} \quad \vec{\epsilon}_f\cdot\vec{\epsilon}_0$$

2 points for Bessel fun. J_1

$$A = ika^2\,\frac{J_1(ka\sin\theta)}{ka\sin\theta}\,\hat{\epsilon}_f\cdot\hat{\epsilon}_0$$

which is the same as Eq. (12.64).

(10 points) for part (b)

Pr 11 - 3 : Exercise 12.3 .2

In[201]:= E32
E32b
E32c

Out[201]=

Exercise 12.3.2. Consider high-energy scattering of electromagnetic plane waves (initial direction $\hat{k}_0$, initial polarization vector $\hat{\epsilon}_0$) off of a perfect conductor in the shape of a flat disk of radius a. Consider only the special case of $\hat{k}_0$ perpendicular to the plane of the disk, as shown in Figure 12.6:

(a) Show that the *illuminated* side unpolarized differential cross section is ($\hat{k}_0 \cdot \hat{k} = \cos\theta$)

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{ill}} = \frac{a^2 J_1^2(ka \sin\theta)}{\sin^2\theta} \sin^4(\theta/2),$$

Out[202]=

where $J_1(x)$ is an integer Bessel function.

(b) Plot this for increasingly large values of $ka \gg 1$. Notice the cross section peaks in the backward direction, as one would expect.

Out[203]=

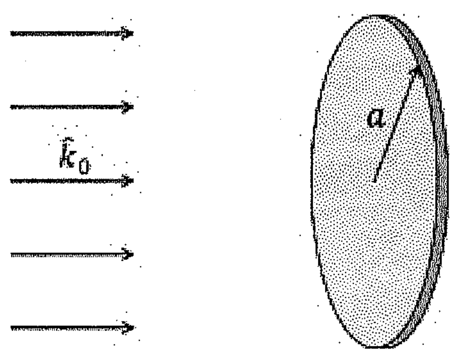
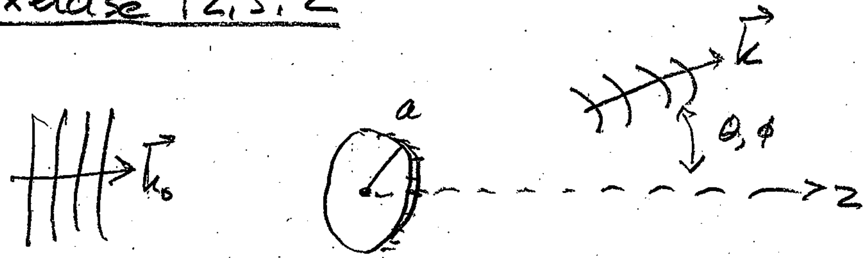


Fig. 12.6 Reference figure for Exercise 12.3.2.

(12 points total)

Exercise 12.3.2



First calculate the polarized cross section, from the illuminated side, by Eq. (12.68).

$$\hat\epsilon_f \cdot \vec f_{ill} = \frac{k}{4\pi i}\int_{ill} da'\, e^{-i(\vec k - \vec k_0)\cdot \vec x'}\, \hat\epsilon_f \cdot$$

$$\left[(\hat k - \hat k_0)\times(\hat n' \times \hat\epsilon_0) - \hat k_0 (\hat n' \cdot \hat\epsilon_0)\right]$$

Let the z axis be along $\vec k_0$.

$\vec k_0 = k\,\hat e_z$ and $\hat k_0 = \hat e_z$.

$\hat n' =$ outward normal $= -\hat e_z$

$$[\ldots] = (\hat k - \hat k_0)\times(\hat n' \times \hat\epsilon_0) - \hat k_0 (\hat n' \cdot \hat\epsilon_0)$$

$$= \hat n' (\hat k - \underbrace{\hat k_0)\cdot\hat\epsilon_0}_{\text{zero}} - \hat\epsilon_0 (\hat k - \hat k_0)\cdot\hat n' - \hat k_0 (\hat n' \cdot \hat\epsilon_0)$$

$$= -\hat e_z (\hat k \cdot \hat\epsilon_0) + \hat\epsilon_0 \left[(\hat k - \hat e_z)\cdot \hat e_z\right] + \hat e_z (\hat e_z \cdot \hat\epsilon_0)$$

$$= \hat e_z \left[-\hat k \cdot \hat\epsilon_0 + \hat e_z \cdot \hat\epsilon_0\right] + \hat\epsilon_0 \left[(\hat k - \hat e_z)\cdot \hat e_z\right]$$

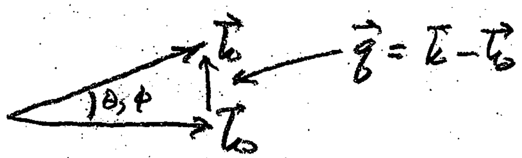
$$= -\hat e_z \left[(\hat k - \hat e_z)\cdot \hat\epsilon_0\right] + \hat\epsilon_0 \left[(\hat k - \hat e_z)\cdot \hat e_z\right]$$

$$= -(\hat k - \hat e_z)\times(\hat e_z \times \hat\epsilon_0)$$

$$\hat{e}_f \cdot \vec{f}_{M} = \boxed{\frac{k}{4\pi i} \, \hat{\epsilon}_f \cdot \left[(\hat{k} - \hat{e}_z) \times (\hat{e}_z \times \hat{\epsilon}_0) \right] (-) \int_{A} da' \, e^{-i(\vec{k}-\vec{k}_0)\cdot \vec{x}'}} \quad \text{(4 points)}$$

The integral

$$= \int_0^a \rho' d\rho' \int_0^{2\pi} d\phi' \; e^{-i(\vec{k}-\vec{k}_0)\cdot \hat{e}_\rho' \rho'}$$



$$\vec{k} = k\{\underset{(x)}{\sin\theta\cos\phi}, \underset{(y)}{\sin\theta\sin\phi}, \underset{(z)}{\cos\theta}\}$$

$$\vec{k}_0 = k\{0, 0, 1\}$$

$$\vec{q} = \vec{k} - \vec{k}_0 = k\{\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta - 1\}$$

$$\vec{x}' = \rho'\{\cos\phi', \sin\phi', 0\}$$

$$\vec{q}\cdot\vec{x}' = k\rho'[\sin\theta\cos(\phi-\phi')]$$

$$\therefore \text{The integral} = \int_0^a \rho' d\rho' \underbrace{\int_0^{2\pi} d\phi' \, e^{-ik\rho'\sin\theta\cos(\phi-\phi')}}_{2\pi J_0(k\rho'\sin\theta)}$$

Bessel Function

$x = k\rho'\sin\theta$

$$= 2\pi \int_0^{ka\sin\theta} \frac{x\,dx\,J_0(x)}{(k\sin\theta)^2}$$

$$J_0(x) = \frac{1}{x}\frac{d}{dx}(x J_1(x))$$

$$\int_0^T dx\, x J_0(x) = T J_1(T)$$

$$= 2\pi \frac{ka\sin\theta \, J_1(ka\sin\theta)}{(k\sin\theta)^2}$$

$$= \underbrace{2\pi a^2 \frac{J_1(ka\sin\theta)}{ka\sin\theta}} \quad \text{(2 points)}$$

Amplitude $A_{fi} = \hat{\epsilon}_f \cdot \vec{f}_M = \frac{ik}{4\pi} \frac{2\pi a J_1(ka\sin\theta)}{k\sin\theta} \hat{\epsilon}_f \cdot \left[(\hat{k} - \hat{e}_z) \times (\hat{e}_z \times \hat{\epsilon}_0) \right]$

Square sum and average $\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{unpol.}$

$$\frac{d\sigma}{d\Omega} = \left(\frac{a}{2}\right)^2 \frac{J_1^2(ka\sin\theta)}{\sin^2\theta} \underbrace{\frac{1}{2}\sum_{f,i}\left\{\hat{\epsilon}_f^* \cdot \left[(\hat{k}-\hat{e}_z)\times(\hat{e}_z\times\hat{\epsilon}_0)\right]\right\}^2}_{\text{calculate this with Mathematica;}}$$

(2 points) $= 4\sin^4\frac{\theta}{2}$ $\quad (1-\cos\theta)^2$

$$\frac{d\sigma}{d\Omega} = \underbrace{a^2 J_1^2(ka\sin\theta)\frac{\sin^4(\theta/2)}{\sin^2\theta}}$$ ← answer in the book

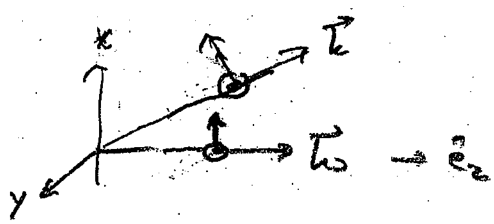
ⓑ Graphical analysis

$\frac{d\sigma}{d\Omega}$ versus θ for $ka = 1, 2, 5, 10, 50$

Needs to use a logarithmic scale because

$\frac{d\sigma}{d\Omega}(\theta=\pi) = a^2 \frac{(ka)^2}{4} \to \infty$ as $ka \to \infty$

(4 points)



Part (a) = 8 points
Part (b) = 4 points

Mathematica claculations for Exercise 12.3 .2

part a

In[173]:=
```
Remove["Global`*"]
ez = {0, 0, 1};
ϵi[1] = {1, 0, 0};
ϵi[2] = {0, 1, 0};
k = {sθ, 0, cθ};
ϵf[1] = {-cθ, 0, sθ};
ϵf[2] = {0, 1, 0};
Dot[k, ϵf[1]] // FullSimplify
Dot[k, ϵf[2]] // FullSimplify
```

Out[180]= 0

Out[181]= 0

In[185]:=
```
polsum = 1/2 * Sum[Sum[
      (Dot[ϵf[j], Cross[k - ez, Cross[ez, ϵi[i]]]]) ^2,
      {j, 1, 2}], {i, 1, 2}];
polsum = polsum /.  {sθ → Sqrt[1 - cθ^2]};
polsum = polsum // FullSimplify
```

Out[187]= $(-1 + c\theta)^2$

part b

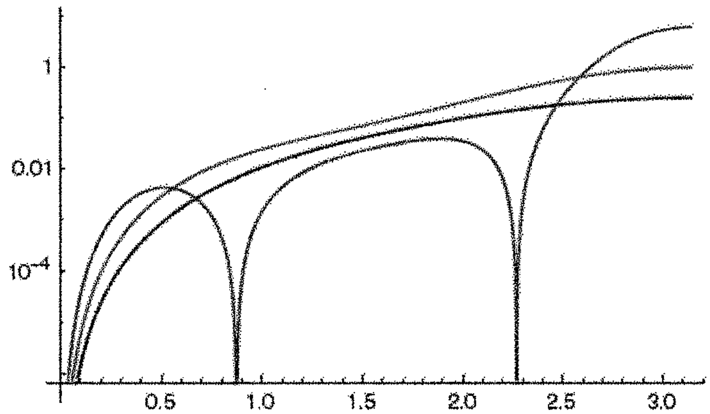
In[190]:=
```
σ[x_, θ_] = BesselJ[1, x * Sin[θ]] ^2 * Sin[θ/2] ^4/Sin[θ] ^2
```

Out[190]= $\text{BesselJ}[1, x \,\text{Sin}[\theta]]^2 \,\text{Csc}[\theta]^2 \,\text{Sin}\left[\frac{\theta}{2}\right]^4$

(3e)

In[191]:=
```
LogPlot[{
   σ[1, θ], σ[2, θ], σ[5, θ]},
  {θ, 0, Pi}] // Rasterize
```

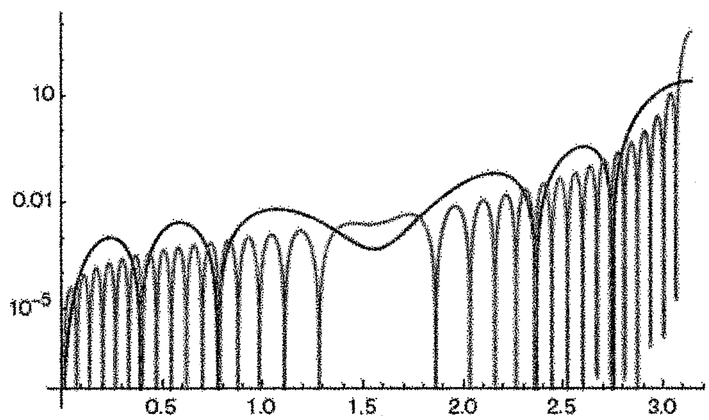
Out[191]=



$ka = 1, 2, 5$

In[192]:=
```
LogPlot[{
   σ[10, θ], σ[50, θ]},
  {θ, 0, Pi}, PlotPoints → 1000] // Rasterize
```

Out[192]=



$ka = 10, 50$

Labels : 2pts

2+ lines : 2pts