H. Assignment 12

Problem 12.1 - Prove that Equation 4.13 is a solution of Bessel's equation

Equation (4.13)

$J_n(t) = i^{-n} \int_0^{2\pi} \frac{d\phi}{2\pi} \exp\{ i\,(t \cos\phi - n\phi)\, \}$

We have to prove that $J_n(t)$ obeys Bessel's equation,

$t^2 J''(t) + t J'(t) + (t^2 - n^2)\, J(t) = 0.$

So, calculate $\left(t^2 \frac{d^2}{dt^2} + t \frac{d}{dt} + t^2 - n^2\right) J_n(t)$

$= i^{-n} \int_0^{2\pi} \frac{d\phi}{2\pi} \exp\{ i\,(t \cos\phi - n\phi)\, \}\ \cdot$

$\cdot\ (\ t^2 \{ (i \cos\phi)^2 \quad + it \cos\phi + t^2 - n^2\ )$

$= i^{-n} \int_0^{2\pi} \frac{d\phi}{2\pi} \exp\{ i\,(t \cos\phi - n\phi)\, \}\ \cdot$

$\cdot\ (\ t^2 \sin^2\phi + it \cos\phi) - n^2\ )$

In the final term, write $-n^2 e^{-in\phi} = \frac{\partial^2}{\partial\phi^2} e^{-in\phi}$; then integrate by parts twice;

$\Longrightarrow e^{-in\phi} \frac{\partial^2}{\partial\phi^2} e^{itcos\phi} = e^{i(tcos\phi - n\phi)} \cdot \{ (-itsin\phi)^2 - it\cos\phi \} = e^{i(tcos\phi - n\phi)} \cdot \{$
$-t^2 \sin^2\phi - it\cos\phi \}$

Therefore, $\left(t^2 \frac{d^2}{dt^2} + t \frac{d}{dt} + t^2 - n^2\right) J_n(t) = 0$ \ Q.E.D.

Problem 12.2

Exercise 12.7.1. Use (12.179) to evaluate and numerically plot the transmission coefficient T as a function of ka in the range $0 < ka < 10$ for the two cases of normal incidence Dirichlet and Neumann scalar circular diffraction in the Fraunhofer limit.

Equation (12.179)

$$\frac{dT}{d\Omega} = \frac{F^2}{\pi \cos\alpha} \left(\frac{J1(ka\Delta)}{\Delta} \right)^2$$

where $\Delta^2 = \sin^2\theta + \sin^2\alpha - 2\sin\alpha\sin\theta\cos\phi$

and $F = \begin{cases} \cos\phi & [\text{Dirichlet } G.F.] \\ -1 & [\text{Neumann } G.F.] \end{cases}$

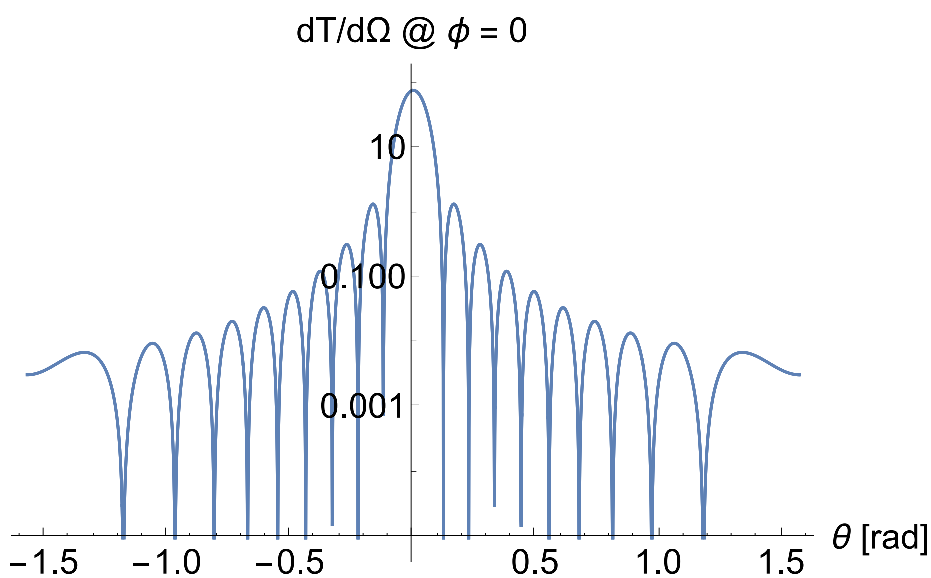
Plot some examples

■ Normal incidence ($\alpha = 0$) with ka = 10π;

In[∘]:=
```
ka = 10 * Pi;
F[ph_] = {Cos[ph], -1};
Δ[th_, ph_, al_] = Sqrt[Sin[th] ^ 2 + Sin[al] ^ 2 -
    2 * Sin[al] * Sin[th] * Cos[ph]];
dT[th_, ph_, al_] = F[ph] ^ 2 / (π * Cos[al]) *
   Power[BesselJ[1, ka * Δ[th, ph, al]] / Δ[th, ph, al], 2];
```

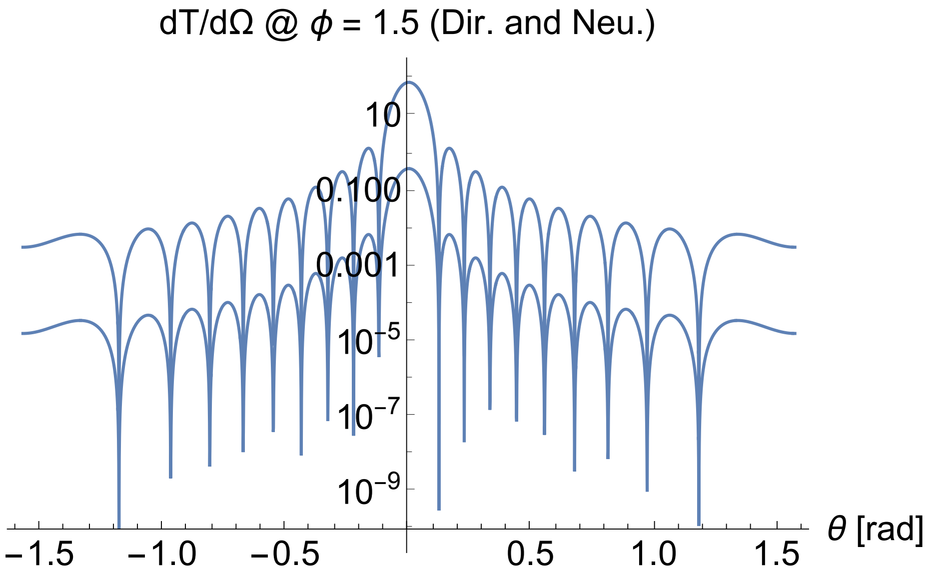
In[∘]:=
```
LogPlot[dT[th, 0, 0], {th, -Pi / 2, Pi / 2},
 AxesLabel → {"θ [rad]", "dT/dΩ @ ϕ = 0"},
 BaseStyle → {18}, ImageSize → 480]
```

Out[∘]=



In[∘]:=
```
LogPlot[dT[th, 1.5, 0], {th, -Pi / 2, Pi / 2},
 AxesLabel → {"θ [rad]", "dT/dΩ @ ϕ = 1.5 (Dir. and Neu.)"},
 BaseStyle → {18}, ImageSize → 480]
```

Out[∘]=

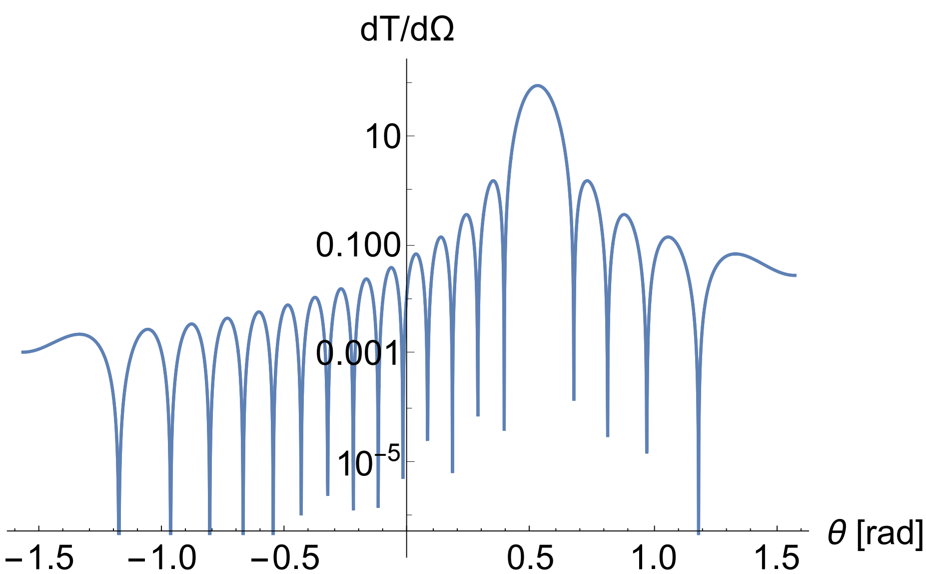


■ Incident at angle 30 degrees (α = Pi/6) with ka = 10π

In[∘]:=
```
3.14 / 6
LogPlot[dT[th, 0, Pi / 6], {th, -Pi / 2, Pi / 2},
 AxesLabel → {"θ [rad]", "dT/dΩ"},
 BaseStyle → {18}, ImageSize → 480]
```

Out[∘]= 0.523333

Out[∘]=



Problem 12.3

Problem 3 Use graphical analysis to compare scalar and vector diffraction for a circular aperture at non normal incidence; see Eqs. (12.179) and (12.216).

The case of scalar diffraction is shown in Problem 12.2.

Equation (12.216) (vector diffraction)

$$\frac{dT}{d\Omega} = \frac{F^2}{\pi \cos\alpha} \left(\frac{J1(ka\Delta)}{\Delta}\right)^2$$

where $\Delta^2 = \sin^2\theta + \sin^2\alpha - 2\sin\alpha \sin\theta \cos\phi$

and $\vec{F} = \begin{cases} \cos\alpha\ (\hat{n} \times \hat{e}_y) & [B_{perp} \text{ polarization}] \\ \hat{n} \times \hat{e}_x & [E_{perp} \text{ polarization}] \end{cases}$

Plot some examples

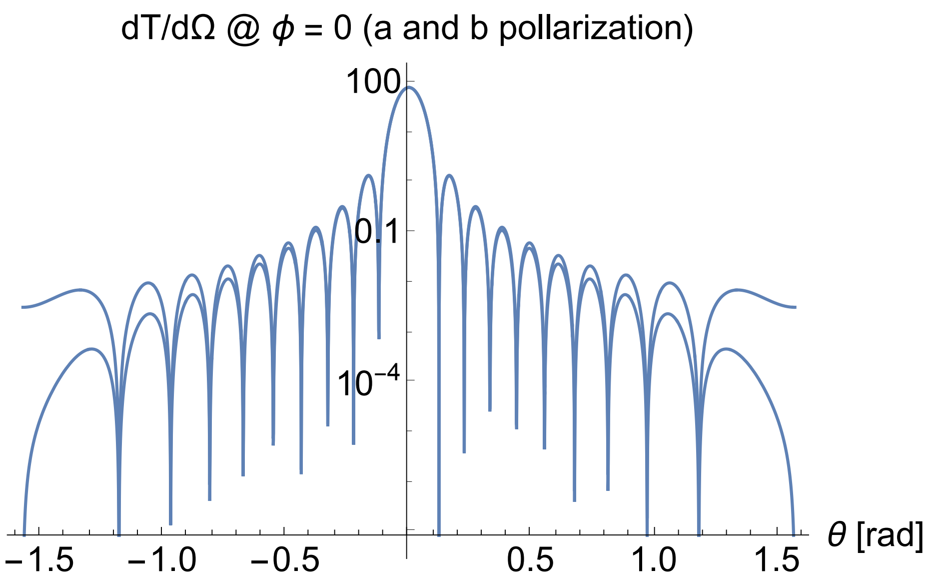
■ Normal incidence ($\alpha = 0$) with ka = 10π;

```
In[∘]:= (* Polarizations *)
ex = {1, 0, 0};
ey = {0, 1, 0};
nh = {Sin[th] Cos[ph], Sin[th] Sin[ph], Cos[th]};
Fasq[th_, ph_, al_] = Cos[al]^2 * Dot[Cross[nh, ey], Cross[nh, ey]];
Fbsq[th_, ph_, al_] = Dot[Cross[nh, ex], Cross[nh, ex]];
```

```
In[∘]:= ka = 10 * Pi;
Fsq[th_, ph_, al_] = {Fasq[th, ph, al], Fbsq[th, ph, al]};
Δ[th_, ph_, al_] = Sqrt[Sin[th]^2 + Sin[al]^2 -
    2 * Sin[al] * Sin[th] * Cos[ph]];
dT[th_, ph_, al_] = Fsq[th, ph, al] / (π * Cos[al]) *
   Power[BesselJ[1, ka * Δ[th, ph, al]] / Δ[th, ph, al], 2];
```

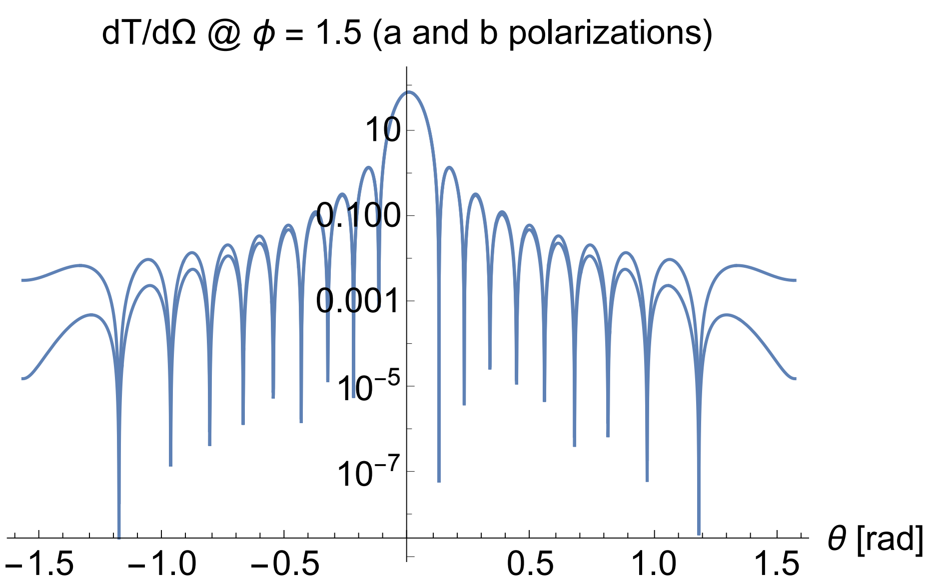
In[∘]:=
```
LogPlot[dT[th, 0, 0], {th, -Pi / 2, Pi / 2},
 AxesLabel → {"θ [rad]", "dT/dΩ @ ϕ = 0 (a and b pollarization)"},
 BaseStyle → {18}, ImageSize → 480]
```

Out[∘]=



In[∘]:=
```
LogPlot[dT[th, 1.5, 0], {th, -Pi / 2, Pi / 2},
 AxesLabel → {"θ [rad]", "dT/dΩ @ ϕ = 1.5 (a and b polarizations)"},
 BaseStyle → {18}, ImageSize → 480]
```

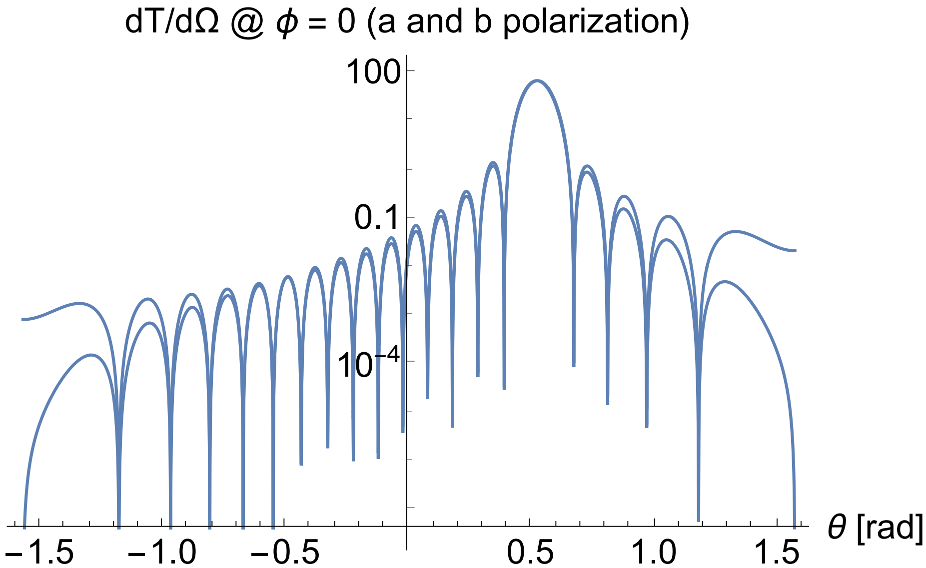
Out[∘]=



▪ Incident waves at a 30 degree angle (α = Pi/6) with ka = 10π;

In[∘]:=
```
LogPlot[dT[th, 0, Pi / 6], {th, -Pi / 2, Pi / 2},
 AxesLabel → {"θ [rad]", "dT/dΩ @ ϕ = 0 (a and b polarization)"},
 BaseStyle → {18}, ImageSize → 480]
```

Out[∘]=



In[∘]:=
```
LogPlot[dT[th, 1.5, Pi / 6], {th, -Pi / 2, Pi / 2},
 AxesLabel → {"θ [rad]", "dT/dΩ @ ϕ = 1.5 (a and b polarization)"},
 BaseStyle → {18}, ImageSize → 480]
```

Out[∘]=