Problem 13-1

(A) Exercise 12.7.3

Given Eqs (12.193) and (12.194) derive the Fraunhofer diffraction limit for single slit diffraction.

In[743]:=

Introducing the *Fresnel number* (see (12.163))

$$N_F \equiv \frac{kd^2}{8\pi\rho}, \qquad (12.192)$$

allows us to simplify the result for the field to:

$$\Phi(\vec{x}) = \frac{iFe^{ikr}}{\sqrt{2}}\Phi_0 e^{i\pi/4} e^{-i\frac{k\rho}{2}\sin^2\phi} \times \left[C\left(\sqrt{2N_F}\left(\frac{2x}{d}+1\right)\right) - C\left(\sqrt{2N_F}\left(\frac{2x}{d}-1\right)\right) + iS\left(\sqrt{2N_F}\left(\frac{2x}{d}+1\right)\right) - iS\left(\sqrt{2N_F}\left(\frac{2x}{d}-1\right)\right)\right], \qquad (12.193)$$

where $x = \rho\sin\varphi$. The angular differential transition rate is defined by:

$$\frac{dT}{d\phi} \equiv \frac{\rho}{d\,|\Phi_0|^2}|\Phi(\vec{x})|^2. \qquad (12.194)$$

Out[743]=

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In[744]:=
```
Remove["Global`*"]
Φ = F * Φ0 / Sqrt[2] *
   ((Cp - Cm) + I * (Sp - Sm))
dTdϕ = ρ / (d * Φ0 ^ 2) * Φ * Conjugate[Φ]
dTdϕ = dTdϕ // ComplexExpand // FullSimplify
```

Out[745]= $$\frac{F\,(-Cm + Cp + i\,(-Sm + Sp))\,\Phi0}{\sqrt{2}}$$

Out[746]= $$\frac{1}{2\,d\,\Phi0}\,F\,(-Cm + Cp + i\,(-Sm + Sp))\,\rho\,(-\text{Conjugate}[Cm] + \text{Conjugate}[Cp + i\,(-Sm + Sp)])\,\text{Conjugate}[F\,\Phi0]$$

Out[747]= $$\frac{F^2\left((Cm - Cp)^2 + (Sm - Sp)^2\right)\rho}{2\,d}$$

In[748]:=
```
Cp = 1 / 2 + 1 / (Pi * zp) * Sin[Pi * zp ^ 2 / 2]
Cm = 1 / 2 + 1 / (Pi * zm) * Sin[Pi * zm ^ 2 / 2]
A = (Cp - Cm) ^ 2 // FullSimplify
```

Out[748]= $$\frac{1}{2} + \frac{\text{Sin}\left[\frac{\pi\,zp^2}{2}\right]}{\pi\,zp}$$

Out[749]= $$\frac{1}{2} + \frac{\text{Sin}\left[\frac{\pi\,zm^2}{2}\right]}{\pi\,zm}$$

Out[750]= $$\frac{\left(zp\,\text{Sin}\left[\frac{\pi\,zm^2}{2}\right] - zm\,\text{Sin}\left[\frac{\pi\,zp^2}{2}\right]\right)^2}{\pi^2\,zm^2\,zp^2}$$

In[751]:=
```
Sp = 1 / 2 - 1 / (Pi * zp) * Cos[Pi * zp ^ 2 / 2]
Sm = 1 / 2 - 1 / (Pi * zm) * Cos[Pi * zm ^ 2 / 2]
B = (Sp - Sm) ^ 2 // FullSimplify
```

Out[751]= $$\frac{1}{2} - \frac{\text{Cos}\left[\frac{\pi\,zp^2}{2}\right]}{\pi\,zp}$$

Out[752]= $$\frac{1}{2} - \frac{\text{Cos}\left[\frac{\pi\,zm^2}{2}\right]}{\pi\,zm}$$

Out[753]= $$\frac{\left(zp\,\text{Cos}\left[\frac{\pi\,zm^2}{2}\right] - zm\,\text{Cos}\left[\frac{\pi\,zp^2}{2}\right]\right)^2}{\pi^2\,zm^2\,zp^2}$$

In[754]:=
```
τ = A + B
τ = τ // TrigExpand // FullSimplify
```

Out[754]=
$$\frac{\left(zp\,\text{Cos}\left[\frac{\pi\,zm^2}{2}\right] - zm\,\text{Cos}\left[\frac{\pi\,zp^2}{2}\right]\right)^2}{\pi^2\,zm^2\,zp^2} + \frac{\left(zp\,\text{Sin}\left[\frac{\pi\,zm^2}{2}\right] - zm\,\text{Sin}\left[\frac{\pi\,zp^2}{2}\right]\right)^2}{\pi^2\,zm^2\,zp^2}$$

Out[755]=
$$\frac{zm^2 + zp^2 - 2\,zm\,zp\,\text{Cos}\left[\frac{1}{2}\,\pi\,(zm - zp)\,(zm + zp)\right]}{\pi^2\,zm^2\,zp^2}$$

In[756]:=
```
τ1 = τ /. {zp → α + β, zm → α - β}
τ1 = τ1 // Expand // FullSimplify
τ2 = τ1 /. {α → cc * 2 x / d, β → cc * (1)}
τ2 = τ2 // Expand // FullSimplify
```

Out[756]=
$$\frac{(\alpha - \beta)^2 + (\alpha + \beta)^2 - 2\,(\alpha - \beta)\,(\alpha + \beta)\,\text{Cos}[2\,\pi\,\alpha\,\beta]}{\pi^2\,(\alpha - \beta)^2\,(\alpha + \beta)^2}$$

Out[757]=
$$\frac{2\,\left(\alpha^2 + \beta^2 + \left(-\alpha^2 + \beta^2\right)\,\text{Cos}[2\,\pi\,\alpha\,\beta]\right)}{\pi^2\,\left(\alpha^2 - \beta^2\right)^2}$$

Out[758]=
$$\frac{2\,\left(cc^2 + \frac{4\,cc^2\,x^2}{d^2} + \left(cc^2 - \frac{4\,cc^2\,x^2}{d^2}\right)\,\text{Cos}\left[\frac{4\,cc^2\,\pi\,x}{d}\right]\right)}{\pi^2\,\left(-cc^2 + \frac{4\,cc^2\,x^2}{d^2}\right)^2}$$

Out[759]=
$$\frac{2\,d^2\,\left(d^2 + 4\,x^2 + \left(d^2 - 4\,x^2\right)\,\text{Cos}\left[\frac{4\,cc^2\,\pi\,x}{d}\right]\right)}{cc^2\,\pi^2\,\left(d^2 - 4\,x^2\right)^2}$$

```
In[760]:= τ3 = τ2 /. {cc → Sqrt[2*k*d^2/(8 Pi*ρ)]}
τ3 = τ3 // Expand // FullSimplify
τ4 = τ3 /. {x → ρ *sϕ }
τ4 = τ4 // Expand // FullSimplify
```

Out[760]= $$\frac{8\,\rho\left(d^2+4\,x^2+\left(d^2-4\,x^2\right)\,\text{Cos}\left[\frac{d\,k\,x}{\rho}\right]\right)}{k\,\pi\left(d^2-4\,x^2\right)^2}$$

Out[761]= $$\frac{8\,\rho\left(d^2+4\,x^2+\left(d^2-4\,x^2\right)\,\text{Cos}\left[\frac{d\,k\,x}{\rho}\right]\right)}{k\,\pi\left(d^2-4\,x^2\right)^2}$$

Out[762]= $$\frac{8\,\rho\left(d^2+4\,s\phi^2\,\rho^2+\left(d^2-4\,s\phi^2\,\rho^2\right)\,\text{Cos}[d\,k\,s\phi]\right)}{k\,\pi\left(d^2-4\,s\phi^2\,\rho^2\right)^2}$$

Out[763]= $$\frac{8\,\rho\left(d^2+4\,s\phi^2\,\rho^2+\left(d^2-4\,s\phi^2\,\rho^2\right)\,\text{Cos}[d\,k\,s\phi]\right)}{k\,\pi\left(d^2-4\,s\phi^2\,\rho^2\right)^2}$$

```
In[764]:= τ5 = τ4  /.  {ρ  →  1/σ}
τ5 = τ5  // Expand // FullSimplify
τ6 = Series[τ5, {σ, 0, 2}]
τ6 = Normal[τ6]
```

Out[764]= $$\frac{8\left(d^2+\frac{4\,s\phi^2}{\sigma^2}+\left(d^2-\frac{4\,s\phi^2}{\sigma^2}\right)\,\text{Cos}[d\,k\,s\phi]\right)}{k\,\pi\left(d^2-\frac{4\,s\phi^2}{\sigma^2}\right)^2\,\sigma}$$

Out[765]= $$\frac{8\,\sigma\left(4\,s\phi^2+d^2\,\sigma^2+\left(-4\,s\phi^2+d^2\,\sigma^2\right)\,\text{Cos}[d\,k\,s\phi]\right)}{k\,\pi\left(-4\,s\phi^2+d^2\,\sigma^2\right)^2}$$

Out[766]= $$-\frac{2\,(-1+\text{Cos}[d\,k\,s\phi])\,\sigma}{k\,\pi\,s\phi^2}+O[\sigma]^3$$

Out[767]= $$-\frac{2\,\sigma\,(-1+\text{Cos}[d\,k\,s\phi])}{k\,\pi\,s\phi^2}$$

```
In[768]:= ans = F^2/(2 d σ) *τ6
```

Out[768]= $$-\frac{F^2\,(-1+\text{Cos}[d\,k\,s\phi])}{d\,k\,\pi\,s\phi^2}$$

The result is

$$\frac{dT}{d\phi}=\frac{F^2}{dk\pi}\;\frac{1}{\sin^2\phi}\;[\,1-\cos(kd\sin\phi\})\,]$$

$$= \frac{F^2}{dk\pi} \quad \frac{1}{\sin^2\phi} \, 2\sin^2\!\left(\frac{1}{2}\,\mathrm{kdsin}\phi\right)$$

$$\frac{dT}{d\phi} = \frac{2F^2}{\pi\,\mathrm{kd}} \left[\frac{\sin^2(1/2\,\mathrm{kd}\sin\phi)}{\sin^2\phi} \right]$$

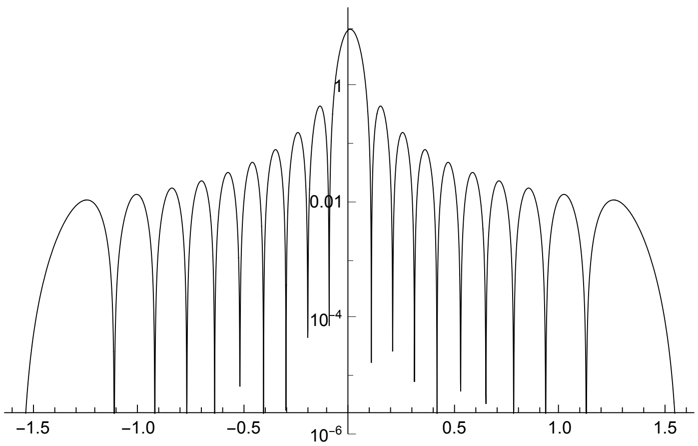
which is equation 12.197.

(B) Plot dT/dϕ

In[784]:=
```
kd = 2 Pi / (0.1 d) * d
f[ϕ_] := 2 / (Pi * kd) * Power[
   Sin[1 / 2 * kd * Sin[ϕ]] / Sin[ϕ], 2]
LogPlot[f[ph], {ph, -Pi / 2, Pi / 2}]
```

Out[784]= 62.8319

Out[786]=



Problem 13-2

Jackson Exercise 13.9

Problem 13-3

(A) HAWC

▮High Altitude Water Cherenkov obsertatory;
▮Cosmic photons of order > TeV;
▮Primary photon starts a electromagnetic shower in the atmosphere and the charged particles produce Cherenkov radiation in the water;
▮discovered sources of high energy photons

(B) T2K

▮Super Kamioka-nde detector;
▮seeking neutrino oscillations;
▮reactor neutrino + H2O → electron, and the electron creates Cherenkov radiation in the water;
▮discovered neutrino oscillations

(C) Ice Cube

▮Neutrino astronomy
▮high energy neutrino+H2O → electron or muon or tau
▮the charged particle has high energy ⇒ Cherenkov radiation
▮discovered very high energy cosmic neutrinos

Problem 13-4

We have a charged particle with constant velocity $\vec{v} = v_0\,\hat{e}_x$ in free space.

(A) Calculate the fields

The potentials are

$\Phi = \gamma\, e\,/s$ and $\vec{A} = \vec{\beta}\,\Phi$ where s = $[\,\gamma^2(x - v_0\,t)^2 + y^2 + z^2\,]^{1/2}$.

In[899]:=
```
Remove["Global`*"]
s = Sqrt[γ^2*(x - β*c*t)^2 + y^2 + z^2];
Φ = γ/s;
A = {β*Φ, 0, 0};
gradΦ = {D[Φ, x], D[Φ, y], D[Φ, z]};
Adot = D[A, t];
El = - gradΦ - 1/c*Adot;
El = El // Simplify;
Print["E field =", El]
B = Curl[A, {x, y, z}];
Print["B field =", B]
```

E field $=\left\{-\frac{(x - c\,t\,\beta)\,(-1+\beta^2)\,\gamma^3}{\left(y^2+z^2+(x-c\,t\,\beta)^2\,\gamma^2\right)^{3/2}},\ \frac{y\,\gamma}{\left(y^2+z^2+(x-c\,t\,\beta)^2\,\gamma^2\right)^{3/2}},\ \frac{z\,\gamma}{\left(y^2+z^2+(x-c\,t\,\beta)^2\,\gamma^2\right)^{3/2}}\right\}$

B field $=\left\{0,\ -\frac{z\,\beta\,\gamma}{\left(y^2+z^2+(x-c\,t\,\beta)^2\,\gamma^2\right)^{3/2}},\ \frac{y\,\beta\,\gamma}{\left(y^2+z^2+(x-c\,t\,\beta)^2\,\gamma^2\right)^{3/2}}\right\}$

(B) The fields move with velocity $\vec{v}$.

Let E(x, y, z, 0) = f(x , y , z)
Note that E(x, y, z, t) = $f(x - v_0 t$, y , z), so the electric field moves with velocity $v_0\,\hat{e}_x$, along with the particle.

(C) A drawing with electric field vectors and retarded-time positions

$t_r = t - R/c = t - |\,\vec{x} - \vec{v}\,t_r\,|\,/c$

For point P_2, t = 0 and $\vec{x}$ = { 0,0,d } and $\vec{\beta}$ = { 0.9,0,0}.

$c\ t_r = -\ \text{Sqrt}[\ d^2 + (0.9\ c\ t_r)^{\ 2}\ ]$
$c\ t_r = \pm d/\text{Sqrt}[1 - 0.9^2] = -\ \gamma d$
$x_r = \beta c t_r = -\beta\gamma d$ and $y_r = z_r = 0$.

For point P_1, t=0 and $\vec{x} = \{\ 0.7, 0, 0.7\}$ d and $\vec{\beta} = \{\ 0.9, 0, 0\}$.

In[1075]:=
```
(* calculations *)
β = 0.9; vel = {β, 0, 0}; γ = 1 / Sqrt[1 - β ^ 2];
μ = 1 / Sqrt[2]; pos1 = {μ, 0, μ};
eqn = {tr == - Sqrt[Dot[pos1 - vel * tr, pos1 - vel * tr]]};
Solve[eqn, tr]
x1r = 0.9 * tr /. %[[1]]
pos2 = {0, 0, 1};
eqn = {tr == - Sqrt[Dot[pos2 - vel * tr, pos2 - vel * tr]]};
Solve[eqn, tr]
x2r = 0.9 * tr /. %[[1]]
```

Out[1078]= {{tr → -7.40926}}

Out[1079]= -6.66833

Out[1082]= {{tr → -2.29416}}

Out[1083]= -2.06474

In[948]:=
```
(* Point P1 *)
β = 0.9; γ = Power[1 - β ^ 2, -1 / 2];
EP1 = El /. { t → 0, x → μ, y -> 0, z → μ}
(* Point P2 *)
EP2 = El /. { t → 0, x → 0, y -> 0, z → 1}
```

Out[949]= {0.292727, 0., 0.292727}

Out[950]= {0., 0., 2.29416}

In[1092]:=
```
a = Plot[0, {x, -10, 10}, PlotRange → {{-7, 3}, {-1, 9}},
    AspectRatio → 0.7];
b = {Red, PointSize[0.03], Point[{-2.065, 0}]};
c = {Red, PointSize[0.03], Point[{-6.668, 0}]};
d1 = {Black, PointSize[0.025], Point[{0, 1}]};
d = {Thickness[0.005], Arrow[{{0, 1}, {0, 3 * 2.294}}]};
e1 = {Black, PointSize[0.025], Point[{μ, μ}]};
e = {Thickness[0.005],
    Arrow[{{μ, μ}, {μ + 3 * 0.293, μ + 3 * 0.293}}]};
Show[a, Graphics[{b, c, d1, d, e1, e}],
 AxesLabel → {"x", "z"}, Ticks → {None, None},
 BaseStyle → {18},
 PlotLabel → "E field vectors at time t = 0"]
```

Out[1099]=