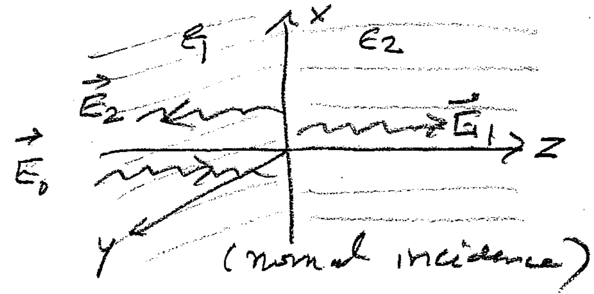
Problem 14.2



(A) Wave propagation in a dielectric

$$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \quad \text{and} \quad \nabla\times\vec{H} = \cancel{\frac{4\pi}{c}\vec{J}}^{\,0} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$$

$$\vec{H} = \vec{B}/\mu \quad \text{and} \quad \vec{D} = \epsilon\vec{E}$$

Let $\vec{E} = E_0\,\hat{e}_x\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ and $\vec{B} = B_0\,\hat{e}_y\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

and $\vec{k} = k\,\hat{e}_z$.

$$i\vec{k}\times E_0\hat{e}_x = \frac{i\omega}{c}B_0\hat{e}_y \qquad i\vec{k}\times\frac{B_0}{\mu}\hat{e}_y = \frac{-i\omega\epsilon}{c}E_0\hat{e}_x$$

$$kE_0 = \frac{\omega}{c}B_0 \qquad kB_0 = \epsilon\mu\frac{\omega}{c}E_0$$

$$\frac{B_0}{E_0} = \frac{ck}{\omega} = \epsilon\mu\frac{\omega}{ck} \Rightarrow k = \sqrt{\epsilon\mu}\,\frac{\omega}{c} \quad ; \text{ also } \frac{B_0}{E_0} = \frac{ck}{\omega} = \sqrt{\epsilon\mu}$$

Boundary Conditions $E_x^{(1)} = E_x^{(2)}$ (E_{tang} is continuous)

$$\frac{1}{\mu_1}B_y^{(1)} = \frac{1}{\mu_2}B_y^{(2)} \quad (H_{tang} \text{ is continuous})$$

(B) Reflection and refraction, at normal incidence.

$$E_0 + E_2 = E_1$$

$$\frac{1}{\mu_1}\sqrt{\epsilon_1\mu_1}\,(E_0 - E_2) = \frac{1}{\mu_2}\sqrt{\epsilon_2\mu_2}\,E_1$$

$$\Rightarrow \quad E_1 - E_2 = E_0$$

$$\sqrt{\frac{\epsilon_2}{\mu_2}}E_1 + \sqrt{\frac{\epsilon_1}{\mu_1}}E_2 = \sqrt{\frac{\epsilon_1}{\mu_1}}E_0$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ \sqrt{\frac{\epsilon_2\mu_1}{\epsilon_1\mu_2}} & +1 \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} E_0 \\ E_0 \end{bmatrix}$$

Solve for E_1 = the transmitted wave

$$M^{-1} = \frac{1}{1+\sqrt{\frac{\epsilon_2\mu_1}{\epsilon_1\mu_2}}}\begin{bmatrix} 1 & 1 \\ -\sqrt{\frac{\epsilon_2\mu_1}{\epsilon_1\mu_2}} & 1 \end{bmatrix} \Rightarrow E_1 = \frac{2E_0}{1+\sqrt{\frac{\epsilon_2\mu_1}{\epsilon_1\mu_2}}} = \frac{2\sqrt{\epsilon_1\mu_2}\,E_0}{\sqrt{\epsilon_1\mu_2}+\sqrt{\epsilon_2\mu_1}}$$

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{H} = \frac{c}{4\pi\mu}\vec{E}\times\vec{B}$$

$$\langle\vec{S}_0\rangle = \frac{c}{8\pi\mu_1}\sqrt{\epsilon_1\mu_1}\,E_0^2\,\hat{e}_z$$

$$\langle\vec{S}_1\rangle = \frac{c}{8\pi\mu_2}\sqrt{\epsilon_2\mu_2}\,E_1^2\,\hat{e}_z$$

Transmission coefficient $(\mu_1 = \mu_2 = 1)$

$$T = \sqrt{\frac{\epsilon_2\mu_2}{\epsilon_1\mu_1}}\,\frac{E_1^2}{E_0^2} = \frac{4\sqrt{\epsilon_2}\,\epsilon_1}{\sqrt{\epsilon_1}\,(\sqrt{\epsilon_1}+\sqrt{\epsilon_2})^2} = \frac{4\sqrt{\epsilon_1\epsilon_2}}{(\sqrt{\epsilon_1}+\sqrt{\epsilon_2})^2}$$

Problem 14.3 Attenuation of an e.m. wave in a metal.

(A) Maxwell's equations $\nabla\cdot\vec{D}=0$ and $\nabla\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$;

$\nabla\cdot\vec{B}=0$ and $\nabla\times\vec{H}=\frac{4\pi}{c}\vec{J}+\frac{1}{c}\frac{\partial\vec{D}}{\partial t}$.

(B) Say $\epsilon=\mu=1$. Also, $\vec{J}=\sigma\vec{E}$.

$$\nabla\times(\nabla\times\vec{E})=-\nabla^2\vec{E}=-\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{4\pi}{c}\sigma\vec{E}+\frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)$$

(C) Have $e^{i(\vec{k}\cdot\vec{x}-\omega t)} \Rightarrow \nabla=i\vec{k}$ and $\frac{\partial}{\partial t}=-i\omega$

$$\therefore\quad k^2=\frac{i\omega}{c}\frac{4\pi\sigma}{c}+\frac{\omega^2}{c^2}$$

$$k=\frac{\omega}{c}\sqrt{1+\frac{4\pi i\sigma}{\omega}}\qquad\text{where } \vec{k}=k\hat{e}_z$$

$$k=\alpha+i\beta \Rightarrow \alpha^2-\beta^2+2i\alpha\beta=\frac{\omega^2}{c^2}\left[1+\frac{4\pi i\sigma}{\omega}\right]$$

$$\left.\begin{aligned}\alpha^2-\beta^2&=\omega^2/c^2\\ 2\alpha\beta&=\frac{4\pi\sigma\omega}{c^2}\end{aligned}\right\}\quad \text{Solve for } \beta:\quad \alpha^2=\beta^2+\frac{\omega^2}{c^2}=\left[\frac{4\pi\sigma\omega}{c^2 2\beta}\right]^2$$

$$\beta^4+\beta^2\frac{\omega^2}{c^2}=\left[\frac{2\pi\sigma\omega}{c^2}\right]^2$$

$$\left(\beta^2+\frac{\omega^2}{2c^2}\right)^2=\left(\frac{\omega^2}{2c^2}\right)^2+\left(\frac{2\pi\sigma\omega}{c^2}\right)^2$$

$$\beta^2=-\frac{\omega^2}{2c^2}\pm\sqrt{\left(\frac{\omega^2}{2c^2}\right)^2+\left(\frac{2\pi\sigma\omega}{c^2}\right)^2}=\frac{\omega^2}{2c^2}\left[-1+\sqrt{1+\left(\frac{2c^2}{\omega^2}\right)^2\left(\frac{2\pi\sigma\omega}{c^2}\right)^2}\right]$$

$$=\frac{\omega^2}{2c^2}\left[-1+\sqrt{1+\frac{16\pi^2\sigma^2}{\omega^2}}\right]=\beta^2$$

(D) Attenuation Length $\equiv\delta$

$$e^{i\vec{k}\cdot\vec{x}}=e^{i(\alpha+i\beta)z}=e^{i\alpha z}e^{-\beta z}$$

$$\delta=\frac{1}{\beta}=\frac{\sqrt{2}c}{\omega}\left[-1+\sqrt{\left(\frac{4\pi\sigma}{\omega}\right)^2+1}\right]^{-1/2}$$

(E) Numerical values $\Rightarrow$

$$\delta=252\ \text{cm}$$

(F) No

Problem 14.4 Classical electron theory

(A) $m\ddot{x} + \gamma\dot{x} + m\omega_0^2 x = 0$

Solutions are e^{pt} where $mp^2 + \gamma p + m\omega_0^2 = 0$

$$p = -\frac{\gamma}{2m} \pm \sqrt{\left(\frac{\gamma}{2m}\right)^2 - \omega_0^2} \approx -\frac{\gamma}{2m} \pm i\omega_0$$

The solution for the given boundary conditions is

$$x(t) = x_0 e^{-\gamma t/2m}\left[\cos\omega_0 t + \frac{\gamma}{2m\omega_0}\sin(\omega_0 t)\right]$$

$$x(t) \approx x_0 e^{-\gamma t/2m}\cos\omega_0 t = \mathrm{Re}\left(x_0 e^{-\gamma t/2m} e^{-i\omega_0 t}\right)$$

$$\ddot{x}(t) \approx x_0 e^{-\gamma t/2m} e^{-i\omega_0 t}\left(-i\omega_0 - \gamma/2m\right)^2$$

The energy radiated is $\mathcal{E} = \int_0^\infty P(t)\,dt = \int_{-\infty}^\infty \frac{2}{3}\frac{e^2}{c^3}|\ddot{x}|^2\,dt$

(Note: $\ddot{x} = 0$ for $t<0$.)

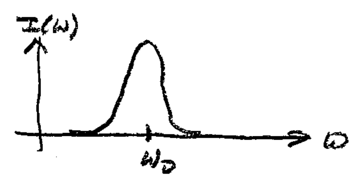
Parseval's Theorem $\int_{-\infty}^\infty |\ddot{x}|^2 dt = \int_{-\infty}^\infty |A(\omega)|^2 d\omega$

where

$$A(\omega) = \int_{-\infty}^\infty e^{i\omega t} a(t)\,dt = \int_0^\infty e^{i\omega t} x_0 e^{-\gamma t/2m} e^{-i\omega_0 t}\left(-i\omega_0 - \frac{\gamma}{2m}\right)^2$$

$$= C \frac{1}{i(\omega-\omega_0) - \gamma/2m}\left[e^{-\gamma t/2m} e^{i(\omega-\omega_0)t}\right]_{t=0}^{\infty} = \frac{-iC}{\omega-\omega_0 + i\gamma/2m}$$

$$|A(\omega)|^2 = \frac{|C|^2}{(\omega-\omega_0)^2 + \left(\frac{\gamma}{2m}\right)^2}$$



$$I(\omega) = \frac{2}{3}\frac{e^2}{c^3}\frac{|C|^2}{(\omega-\omega_0)^2 + (\gamma/2m)^2}$$

(B) $U = U_0 e^{-\Gamma t}$ where $U_0 = \frac{1}{2}m\omega_0^2 x_0^2$

and conservation of energy $\Rightarrow \frac{dU}{dt} = -\langle P\rangle$ average over one cycle

$$\langle P\rangle_{avg} = \frac{2}{3}\frac{e^2}{c^3}\langle(\ddot{x})^2\rangle = \frac{2}{3}\frac{e^2}{c^3}\langle(-\omega_0^2 x_0\cos\omega t)^2\rangle = \frac{2}{3}\frac{e^2}{c^3}x_0^2\omega_0^4 \cdot \frac{1}{2}$$

$$\frac{dU}{dt} = -\Gamma U_0 = -\frac{e^2}{3c^3}x_0^2\omega_0^4 \Rightarrow \Gamma = \frac{e^2 x_0^2\omega_0^4}{3c^3\,\frac{1}{2}m\omega_0^2 x_0^2} = \frac{2e^2\omega_0^2}{3c^3\,m}$$

Problem 14.4 continues

(c) The spectral wavelength is $\lambda = \frac{2\pi c}{\omega_0}$

So $\Gamma = \frac{2}{3}\frac{e^2}{mc^3}\left(\frac{2\pi c}{\lambda}\right)^2 = \frac{8\pi^2}{3}\frac{e^2}{mc\lambda^2}$

The linewidth $\delta\lambda$:

$e^{-\Gamma t} \Rightarrow \delta\omega = \Gamma$

$\omega = \frac{2\pi c}{\lambda}$

$\delta\omega = \frac{-2\pi c}{\lambda^2}\delta\lambda$

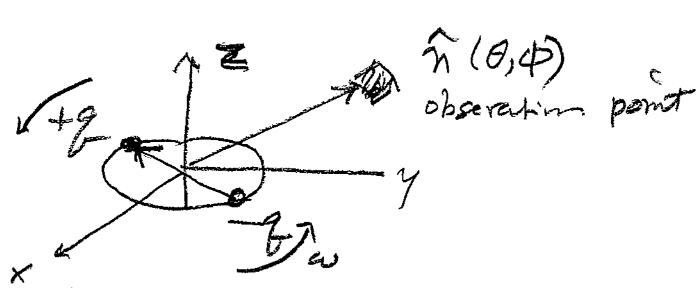
$$\delta\lambda = \frac{\lambda^2}{2\pi c}\Gamma$$

$$= \frac{\lambda^2}{2\pi c}\,\frac{8\pi^2}{3}\frac{e^2}{mc\lambda^2} = \frac{4\pi}{3}\frac{e^2}{mc^2}$$

$$= \frac{4\pi}{3}\,\frac{e^2}{\hbar c}\,\frac{\hbar c}{mc^2}$$

$$= 1.2\times 10^{-13}\ \text{cm} = 1.2\times 10^{-4}\ \text{angstroms}$$

Problem 14-5



(A) There is a time dependent electric dipole moment, so the dominant radiation is electric dipole radiation.

$$\vec{p}(t) = q\vec{d} = qd(\hat{e}_x \cos\omega t + \hat{e}_y \sin\omega t)$$

(B) Vector potential $\vec{A}(\vec{x},t) = \frac{1}{cr}\dot{\vec{p}}(t - r/c)$

Magnetic field $B = \nabla\times\vec{A} = \frac{-1}{cr}\ddot{\vec{p}}\times\left(-\frac{\hat{r}}{c}\right) = \frac{1}{c^2 r}\ddot{\vec{p}}\times\hat{n}$

Energy flux

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = \frac{c}{4\pi}B^2\hat{n} = \frac{c}{4\pi}\frac{(\hat{n}\times\ddot{\vec{p}})^2}{c^4 r^2}\hat{n}$$

$$\therefore \frac{dP}{d\Omega} = r^2\frac{dP}{dA} = \frac{(\hat{n}\times\ddot{\vec{p}})^2}{4\pi c^3}$$

$$\ddot{\vec{p}} = -qd\omega^2(\hat{e}_x\cos\omega t + \hat{e}_y\sin\omega t)$$

$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$(\hat{n}\times\ddot{\vec{p}})^2 = \ddot{\vec{p}}^2 - (\hat{n}\cdot\ddot{\vec{p}})^2$$

$$= (qd\omega^2)^2 - (qd\omega^2)^2(\sin\theta\cos(\phi-\omega t))^2$$

$$= (qd\omega^2)^2[1 - \sin^2\theta\cos^2(\phi-\omega t)]$$

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{(qd\omega^2)^2}{4\pi c^3}\left[1 - \frac{1}{2}\sin^2\theta\right] = \frac{q^2 d^2\omega^4}{8\pi c^3}(1+\cos^2\theta)$$

(average over time)

(C) $$P = \int\left\langle\frac{dP}{d\Omega}\right\rangle d\Omega = \frac{q^2 d^2\omega^4}{8\pi c^3}\cdot\underbrace{\int \sin\theta\, d\theta\, d\phi\,(1+\cos^2\theta)}$$

$$\longrightarrow = 2\pi\int_{-1}^{1} du(1+u^2) = \frac{16\pi}{3}$$

$$P = \frac{2}{3}\frac{q^2 d^2\omega^4}{c^3}$$

(D) Now there are 4 charges (2 real + 2 image) $\Rightarrow$ the electric dipole moment is 0. The dominant radiation will be electric quadrupole.