Problem 15-1

(A) $t' = \gamma\left(t - \frac{v}{c^2}x\right)$

$x' = \gamma(x - vt)$

$y' = y$

$z' = z$

(B) $E'_{\parallel} = E_{\parallel}$

$\vec{E}'_{\perp} = \gamma\left(\vec{E}_{\perp} + \frac{\vec{v}}{c}\times\vec{B}_{\perp}\right)$

$B'_{\parallel} = B_{\parallel}$

$\vec{B}'_{\perp} = \gamma\left(\vec{B}_{\perp} - \frac{\vec{v}}{c}\times\vec{E}_{\perp}\right)$

(C) Transform from $K' \to K$.

$\vec{B}_{\perp} = \gamma\frac{\vec{v}}{c}\times\vec{E}'_{\perp} = \gamma\frac{\vec{v}}{c}\times\frac{q\vec{r}'_{\perp}}{(r')^3}$

For $v \ll c$, $\vec{B} = \frac{q\vec{v}\times\vec{r}}{cr^3}$ (Biot Savart)

Problem 15-2

Note this correction

(A) $\Phi = \frac{\gamma q_1}{s}$ where $s = \sqrt{x^2+y^2+\gamma^2(z-vt)^2}$

$$\frac{1}{c}\frac{\partial\Phi}{\partial t} = \frac{\beta\gamma^3 q_1 (z-vt)}{s^3}$$

$$\vec{A} = \vec{\beta}\,\Phi = \beta\,\hat{e}_z\,\Phi$$

$$\nabla\cdot\vec{A} = \frac{\partial A_z}{\partial z} = \beta\frac{\partial\Phi}{\partial z} = \beta\left(\frac{-1}{v}\right)\frac{\partial\Phi}{\partial t} = -\frac{1}{c}\frac{\partial\Phi}{\partial t}$$

$$\therefore \frac{1}{c}\frac{\partial\Phi}{\partial t} + \nabla\cdot\vec{A} = 0.$$

(B) $\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$

$$E_x = -\frac{\partial\Phi}{\partial x} = \frac{\gamma q_1 x}{s^3} \text{ and } E_y = \frac{\gamma q_1 y}{s^3}$$

$$E_z = -\frac{\partial\Phi}{\partial z} - \frac{1}{c}\frac{\partial}{\partial t}\beta\Phi = \frac{-\partial\Phi}{\partial z} - \frac{\beta}{c}\frac{\partial\Phi}{\partial z}\gamma^2(-v)$$

$$= +(1-\beta^2)\frac{\partial\Phi}{\partial z} = +\frac{1}{\gamma^2}\frac{\gamma^3 q_1(z-vt)}{s^3} = \frac{+\gamma q_1(z-vt)}{s^3}$$

$$\therefore \vec{E} = \frac{\gamma q_1}{s^3}\left[x\hat{e}_x + y\hat{e}_y + (z-vt)\hat{e}_z\right]$$

Problem 15-3 $\qquad m\dot{\vec{v}} = q(\vec{E} + \vec{v}\times\vec{B}) + \frac{2q^3}{3c^3}\ddot{\vec{v}}$

(A)

For $v \ll c$, the radiation reaction is small, so

$$m\dot{\vec{v}} \approx q(\vec{E} + \frac{\vec{v}}{c}\times\vec{B})$$

$$\ddot{\vec{v}} = \frac{q}{m}\dot{\vec{E}} + \frac{q}{mc}\underbrace{\dot{\vec{v}}\times\vec{B}}_{\downarrow\ \frac{q}{m}(E + \frac{\vec{v}}{c}\times\vec{B})\times\vec{B}} + \frac{q}{mc}\vec{v}\times\dot{\vec{B}}$$

↖ neglect because $v \ll c$

$$\ddot{\vec{v}} = \frac{q}{m}\dot{\vec{E}} + \frac{q^2}{m^2c}\vec{E}\times\vec{B}$$

$$\vec{F}_r = \frac{2q^2}{m^2c^3}\left[\frac{q}{m}\dot{\vec{E}} + \frac{q^2}{m^2c}\vec{E}\times\vec{B}\right] \quad \text{for } v \ll c$$

(B) For a polarized plane wave, say $\vec{E} = E_0\hat{e}_x \sin(kz-\omega t)$,

$$\langle\dot{\vec{E}}\rangle = 0 \quad \text{and} \quad \langle\vec{E}\times\vec{B}\rangle = \frac{1}{2}E_0^2\hat{e}_z$$

$$\langle\vec{F}_r\rangle = \frac{e^4}{3m^2c^4}E_0^2\hat{e}_z = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2\frac{E_0^2}{8\pi}\hat{e}_z$$

(a positron is the same)

(C)

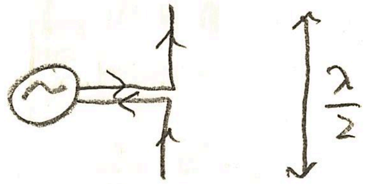
$\langle P\rangle = \langle S_z\rangle\sigma$ where σ = Thomson cross section $= \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2$

Momentum transfer

$$\left\langle\frac{dp_z}{dt}\right\rangle = \left\langle\frac{1}{c}\frac{d\mathcal{E}}{dt}\right\rangle = \frac{E_0^2}{8\pi}\sigma$$

$$\therefore \left\langle\frac{dp_z}{dt}\right\rangle = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2\frac{E_0^2}{8\pi} = \langle F_{rz}\rangle$$

Problem 15-4



(A)

$$\vec{A}(\vec{x},t) = \frac{1}{c}\int_V d^3x' \int dt' \, \frac{\vec{J}(\vec{x}',t')}{|\vec{x}-\vec{x}'|} \, \delta\left(t' - t + \frac{|\vec{x}-\vec{x}'|}{c}\right)$$

$|\vec{x}-\vec{x}'| \approx r$ in the radiation zone

$\vec{J}(\vec{x}',t') = I_0 \hat{e}_z \cos\left(\frac{2\pi z'}{d}\right) \delta(x')\delta(y') e^{i\omega t'}$

$$\vec{A}(\vec{x},t) \approx \frac{I_0 \hat{e}_z}{cr} \int_{-\lambda/4}^{\lambda/4} dz' \cos\left(\frac{2\pi z'}{d}\right) e^{i\omega\left[\underbrace{t - \frac{1}{c}\sqrt{x^2+y^2+(z-z')^2}}_{\approx\, t - \frac{r}{c} + \frac{z'z}{cr}}\right]}$$

in the radiation zone

$$= \frac{I_0 \hat{e}_z}{cr} e^{i\omega(t - r/c)} \int_{-\lambda/4}^{\lambda/4} dz' \cos\left(\frac{2\pi z'}{d}\right) e^{i\frac{\omega}{c}\frac{zz'}{r}}$$

Write $z = r\cos\theta$

$$= \frac{I_0 \hat{e}_z}{cr} e^{i\omega(t-r/c)} \frac{c}{\omega}\left[\frac{\sin\left(\frac{\pi}{2}(1+\cos\theta)\right)}{1+\cos\theta} + \frac{\sin\left(\frac{\pi}{2}(1-\cos\theta)\right)}{1-\cos\theta}\right]$$

$$= \frac{2I_0 \hat{e}_z}{\omega r} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} e^{i(\omega t - kr)} \quad \text{where } k = \frac{\omega}{c}$$

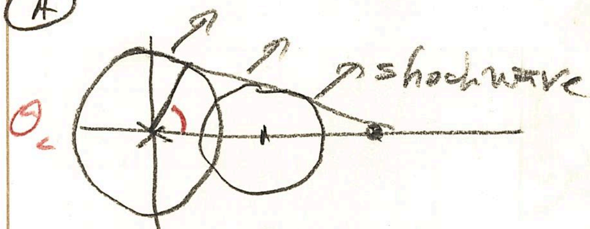
(B) $\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} = \frac{i\omega}{c}\vec{A}$ and $\vec{B} = -ik\,\hat{n}\times\vec{A}$

(C) $$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{c}{8\pi}\frac{\omega k}{c} \mathrm{Re}\left[r^2 \hat{n}\cdot(\vec{A}\times(\hat{n}\times\vec{A}))\right]$$

$$= \frac{I_0^2}{2\pi c} \frac{\cos^2\left(\pi/2 \cos\theta\right)}{\sin^2\theta}$$

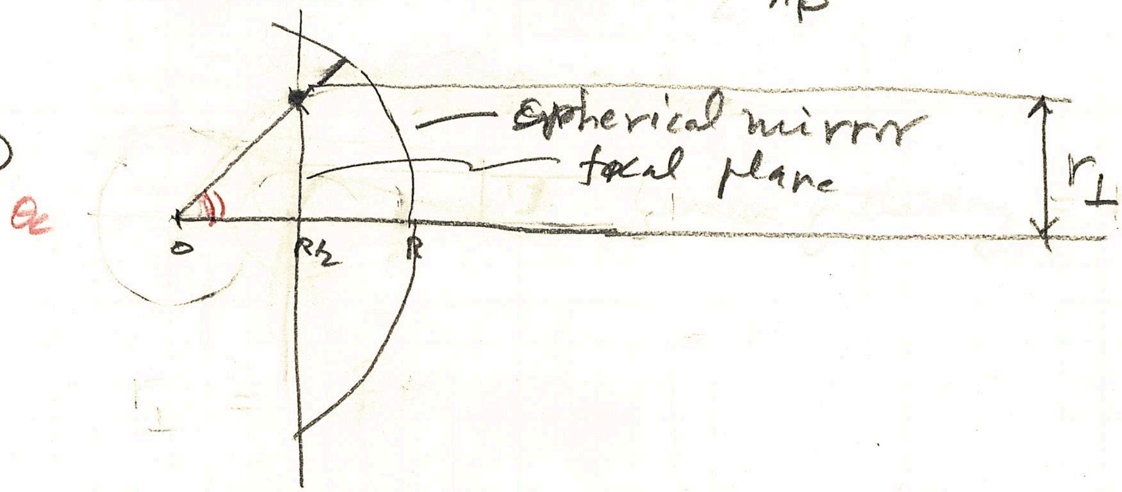
Problem 15-5

(A)



$$\cos\theta_c = \frac{\frac{c}{n}t}{vt} = \frac{1}{n\beta}$$

(B)



radius of the ring is $r_\perp$

$$\tan\theta_c = \frac{r_\perp}{R/2}$$

$$r_\perp = \frac{R}{2}\tan\theta_c \, .$$