THE THEORY OF LIQUID ^{4}He

By J. WILKS
Clarendon Laboratory, Oxford

CONTENTS

Abstract. An account is given of the present state of the theory of liquid ^{4}He. At low temperatures it is no longer possible to consider the motion of individual atoms, and a wave mechanical treatment becomes necessary. Such a treatment, as developed by Landau and Feynman, characterizes the thermal motion of the liquid by 'elementary excitations', the so-called phonons and rotons. A consideration of these excitations then leads to the 'two-fluid' model. It is shown that the treatment is adequate to account for nearly all the properties of the liquid, which are briefly reviewed. The principal problem still outstanding concerns the behaviour of the liquid at high rates of flow or when carrying a large heat current, when it is clear that some form of turbulence is induced.

§ 1. INTRODUCTION

IN this review we shall confine our attention principally to liquid helium II, that is to liquid ^{4}He at temperatures below the so-called lambda point at 2·17°K, which is characterized by an anomaly in the specific heat (figure 1). Above this temperature the liquid is not vastly different from many others ; below this temperature it is unique and exhibits quantum effects on a macroscopic scale. Considerable contributions to the theory have been made by F. London, Tisza, Landau and Feynman ; in particular, Landau in his 1941 paper gave most of the picture as we see it to-day. However, some of his arguments were not presented in any great detail, and as one section of the paper dealing with quantum hydrodynamics is now recognized to be incorrect, the unusual concepts which he introduced did not find immediate acceptance. However, an increasing number of experimental results and theoretical speculations have justified his approach ; also the results he deduced by arguments involving quantum hydrodynamics have been more satisfactorily established by Feynman. This review attempts to set down the present position in a logical manner, both in order to give a general picture to the reader not intimately acquainted with this field, and also to delineate those points where the theory still appears inadequate or in need of modification. In the first four sections we describe the quantum concepts necessary for the treatment of liquid helium, and then in the later sections apply these results to the experimental observations. In order to present as clear a picture as possible, no attempt is made to refer to all the published material, but this is hardly necessary as several comprehensive reviews covering most aspects of the subject are already available (e.g. Dingle 1952 a, Daunt 1952, Wilks 1953, Daunt and Smith 1954). However, in the appendix we give a list of the papers on liquid ^{4}He which have appeared since these reviews were published, and which are not referred to in the present article.

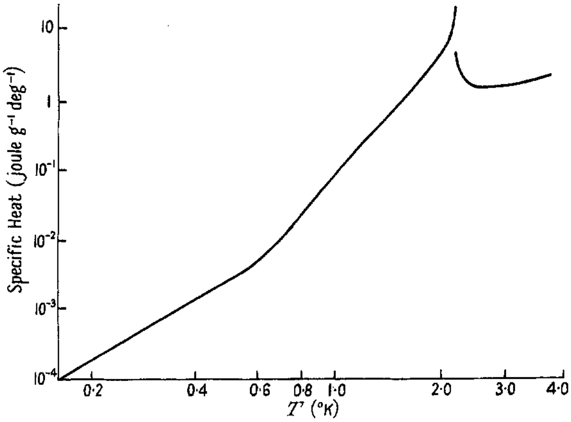

Figure 1. The specific heat of liquid helium (*after* Kramers, Wasscher and Gorter 1952, and Keesom and Clusius 1932).

As may be seen from the melting curve shown in figure 2 (Simon and Swenson 1950), the most important single aspect of liquid helium is that, under its own vapour pressure, it remains a liquid down to absolute zero, and will only solidify under a pressure of at least 25 atmospheres. This behaviour is a consequence of the van der Waals forces being very small; on considering the equilibrium spacing of the atoms in the condensed phase, it is found that the energy associated with the zero-point motion is comparable with the ordinary potential and kinetic-energy terms (Simon 1934). The effect of this zero-point energy is to increase

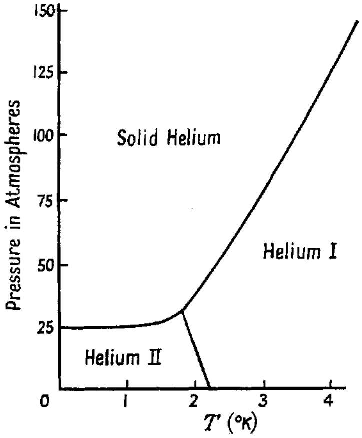

Figure 2. The melting curve of liquid helium. The line between helium I and helium II gives the dependence of the lambda point on pressure (*after* Simon and Swenson 1950).

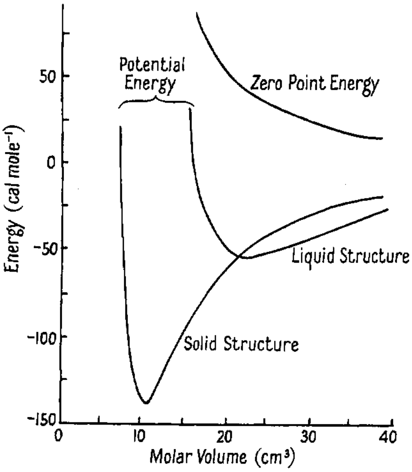

Figure 3. The internal energy of condensed helium at 0°K. The lowest total energy, consisting of the sum of potential and zero-point energy, is obtained with the liquid phase at a molar volume of 27 cm^3 (*after* F. London 1954).

the net kinetic energy, and therefore to increase the atomic spacing to such an extent that the atomic volume is about three times as large (27 cm^3) as would be expected in the absence of zero-point energy. Hence the atoms never come close enough for the van der Waals forces to overlap and form a solid phase. It is difficult to make an exact calculation of the equilibrium condition of a system in which zero-point energy plays a decisive role, but a full discussion of what is possible has been given by F. London (1954). The potential energy of both liquid and solid helium may be calculated from the virial coefficients together with a knowledge of the form of the respective lattices obtained from X-ray analyses (Keesom and Taconis 1938 a, b); the results for temperature zero are shown in figure 3, together with an estimate of the zero-point energy. It can be seen that the total energy at absolute zero, which consists only of potential plus zero-point energy, is least for liquid with a molar volume of about 27 cm^3; the liquid at this volume has less energy than the solid and is therefore the stable phase.

§ 2. PHONONS

Given that the condensed phase of helium at absolute zero is a liquid, we consider what thermal motions are introduced as the temperature is raised. It is an experimental fact that the liquid will transmit longitudinal sound waves (although not transverse ones on account of its very low viscosity), and one therefore concludes that part of the thermal motion is an assembly of longitudinal sound waves. Obviously, since the helium is a liquid and not a solid, other modes of motion are possible, and one of the great difficulties in the theory of liquid helium has been to know how to deal with these other motions. However, it now seems that Feynman has given a satisfactory treatment of the problem, and this will be discussed in the next section. For the moment we need only quote the result that the other thermal motions are quantized and that a certain minimum energy is needed to excite them. Thus at the lowest temperatures the heat motion of the liquid is due only to longitudinal vibrations which may be expected to have a Debye-type spectrum.

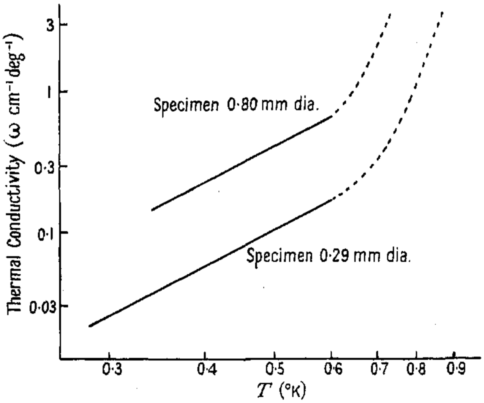

Figure 4. The thermal conductivity of liquid helium at low temperatures. Below 0·6°K the heat flow is proportional to the temperature gradient, and the apparent thermal conductivity varies as the diameter of the specimen (*after* H. A. Fairbank and J. Wilks 1955).

The conclusion that no other modes of motion are permitted, although rather surprising, is well supported by the experimental evidence. Thus below about 0·6°K the specific heat (figure 1) may be written as $0{\cdot}0235T^3$ joules g^{-1} deg^{-1} (Kramers, Wasscher and Gorter 1952), and this is in quite good agreement with the value of $0{\cdot}021T^3$ joules g^{-1} deg^{-1} calculated from the Debye theory using a value of the velocity of sound obtained by extrapolating the results of Atkins and Chase (1951). The ten per cent discrepancy in the numerical factor is of no great significance ; it may be due to some errors in the rather difficult experiments, or perhaps could be explained by a more refined treatment of the sound waves as has been attempted for instance by Lokken (1954). Figure 4 gives values of the thermal conductivity (Fairbank and Wilks 1955) which show that although the behaviour of the liquid is very complex above 0·6°K, below that temperature the liquid behaves in a very similar way to a dielectric solid. The heat flow per unit area is accurately proportional to the temperature gradient, and is given quite closely by the formula of Casimir (1938) for the thermal conductivity of a dielectric solid in which the heat flow is limited only by scattering at the boundary of the specimen. Thus these measurements demonstrate both the presence of the thermal waves and also the absence of any other form of motion capable of scattering these waves in such a way as to produce thermal resistance. Finally the coefficient of thermal expansion, which becomes negative immediately below the lambda point, appears at low temperatures to take up positive values in good agreement with what would be expected if only sound waves were present (Atkins and Edwards 1955).

There is one further point essential to the treatment of the heat motion associated with the longitudinal sound waves : as we are concerned with very low temperatures a quantum treatment is necessary and was first introduced by Landau (1941). He remarks that any weakly excited state can be considered as an aggregate of single elementary excitations which in this particular case are sound quanta, or phonons, with discrete values of energy ϵ and of momentum p related by

$$\epsilon = pc.\dagger \qquad \ldots\ldots(1)$$

The concept of phonons to describe a sound field seems to have been first mentioned by Frenkel (1936) and was justified by analogy with photons. Although it is by no means obvious that the motion of a fluid can be described in this way, Landau did not attempt any further justification of the concept. However, a full investigation has since been made by Kronig and Thellung (1952) and also by F. London (1954), and their conclusions bear out Landau's approach. Kronig and Thellung consider a continuous non-viscous fluid and confine themselves to irrotational motion in the absence of external forces. They first re-state the classical equations of motion for such a liquid, using the velocity potential and the density as canonically conjugate variables, and then carry out a quantization in the normal manner. This leads immediately to the concept of discrete phonons with energy ϵ and momentum p. The treatment also demonstrates the possibility of interactions between the phonons giving rise to scattering and to the creation and annihilation of phonons—processes which had already been postulated and discussed by Landau and Khalatnikov (1949).

§ 3. ROTONS

As is shown in figure 1, the specific heat above 0·6°K becomes rapidly greater than the Debye value, and it is clear that some other forms of thermal motion are being excited. The question, however, is just what form these excitations take. Landau believed that some form of vortex or rotational motion was involved, and to investigate this possibility set up a new formulation of quantum hydrodynamics. He claimed to show that vortex motion is quantized, and that the energy and

† where c is the velocity of sound.

momentum of an elementary excitation or 'roton' are related by the equation

$$\epsilon = \Delta + p^2/2\mu \qquad \ldots\ldots(2)$$

where Δ and μ are constants. This relation shows that a minimum energy Δ is required to create a roton, and immediately leads to the conclusion that at sufficiently low temperature the rotons will not contribute to the thermal energy. At higher temperatures the total thermal energy of the liquid is the energy of an assembly of photons and rotons, and the thermodynamic functions may be calculated by the usual statistical methods. By comparing the expressions for the specific heat obtained in this way with the experimental values, Landau obtained values for Δ/k of 8–9°K (k is Boltzmann's constant), and for μ of 7–8 times the mass of the helium atom. This comparatively large value for Δ correlates with the absence of rotons below about 0·6°K and the rapid rise in the specific heat above that temperature.

Although the concept of a roton with an energy spectrum similar to that of equation (2) has turned out to be very fruitful, Landau's derivation of it is now regarded as unacceptable. It is far from clear that the operators in his quantum-mechanical treatment have the properties he somewhat intuitively claims for them (for detailed criticisms, see F. London 1945 and Dingle 1952 a). Quite recently Ziman (1953 a) made a new attack on the problem rather in the way that Kronig and Thellung (1952) had treated the sound waves ; he was able to set up the classical equations of hydrodynamics in a form suitable for quantization and claimed to show that the Hamiltonian contained terms corresponding to the excitation of a roton state. However, this treatment also appears open to objection, because other terms in the Hamiltonian correspond to interactions which have to be taken into account if more than one roton is excited. It has yet to be shown that the same conclusions will be valid when these interactions are included ; other difficulties have also been pointed out by Allcock and Kuper (1955).

After Peshkov's measurements on temperature waves (to which we refer later), Landau (1947) quite empirically modified the energy spectrum to the form :

$$\epsilon = \Delta + \frac{(p - p_0)^2}{\mu} \qquad \ldots\ldots(3)$$

in order to obtain better agreement with the experimental results. This last equation has turned out to be so useful in interpreting the behaviour of liquid helium that the uncertain foundations on which it was based were rather glossed over until recently, when Feynman gave a derivation of the equation which is far more satisfactory. Feynman also shows, what Landau only tacitly assumed, that the phonon states are the only possible states at low temperatures. (As Landau's hydrodynamical approach used only coarse-grained variables, it did not rule out the possibility of other excitations associated with motion on an atomic scale.) Finally, before giving a summary of Feynman's treatment, it should be pointed out that although Landau's derivation of the roton spectrum is almost certainly incorrect, it was he who made the essential step of postulating that the whole motion of the liquid could be expressed in terms of quantized elementary states. These states are, it must be emphasized, wave-mechanical states of the whole liquid and are characterized by their energy and momentum.

We now give a brief summary of Feynman's approach, referring the reader to the original papers for a full and lucid discussion of the ideas involved (Feynman 1953 a, b, 1954, 1955; Feynman and Cohen 1956). He regards the liquid as an assembly of atoms, between pairs of which there are strong repulsive and weak attractive van der Waals forces, and which to a first approximation may be treated as hard spheres of diameter about $2{\cdot}7A$. As the mean volume per atom is about $(3{\cdot}6A)^3$, the structure is a very open one, and the atoms can move about freely either through holes in the lattice or by pushing other atoms aside. Feynman first discusses density fluctuations, and agrees with previous authors that there will be a continuous range of energy levels right down to the lowest values, corresponding to sound waves of different wavelengths. He next considers what other forms of excitation are possible, and remarks that any such excitations will correspond to various rearrangements of the atoms at constant density. Moreover, because of the very open structure of the liquid, such rearrangements may in principle be brought about by a simple stirring process, and Feynman proceeds to deduce appropriate wave functions to describe the liquid. The wave functions used in his first papers were recognized to be only approximate and led to difficulties concerning the conservation of mass (Feynman 1954), and better functions have been given recently by Feynman and Cohen (1956). However, as the earlier work forms a better introduction to the subject, and as no great difference of principle is involved, we now outline Feynman's theory using his earlier wave functions.

Suppose the helium to be in the rth stationary state with a given energy E_r and let the position of the ith atom be defined by its coordinate $\mathbf{R}_i$; then the wave function for that state may be written $\psi_r(\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_N)$, a definite though complicated function of the coordinates $\mathbf{R}$. As the atoms have a tendency to keep apart, ψ_r will have its greatest values for those configurations in which the atoms are widely separated; and as the atoms are regarded as impenetrable spheres ψ_r must fall to zero for configurations in which atoms overlap. As usual the ground state $\psi = \phi$ is characterized by the absence of nodes, and ϕ may be taken to be a positive number for all configurations of the $\mathbf{R}$'s.

Now consider a wave function ψ_r which is to represent a stationary state of higher energy. ψ_r must be orthogonal to the ground state ϕ, which is everywhere positive; that is, ψ_r must be positive for some configurations $(\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_N)$ and negative for the others. Thus if ψ_r is plotted against the $3N$ coordinates of configuration space it will vary between certain maximum positive and negative values. We wish to find states of low energy, and the criterion for such states is that the gradient of ψ_r in configuration space should be small. (We recall that in the simple problem of a particle in a box, the wave function oscillates more frequently the higher the energy, as the kinetic energy is given by $-(\hbar^2/2m)\nabla^2\psi$.) Let us now refer to the configuration which gives ψ_r its maximum positive value as an A configuration, and that which gives the maximum negative value as a B configuration. Then if ψ_r is to represent a state of low energy, the points corresponding to configurations A and B should be widely separated in configuration space. If the atoms are distinguishable, this condition implies that to move the atoms from an A configuration to a B configuration involves a stirring process in which all the atoms are displaced a long way. However, as all the atoms are indistinguishable and obey Bose statistics, it must be possible to go from an A configuration to a B, or indeed to any other one, by moving each atom by a distance of no more than about the atomic spacing. This means that configurations A and B can only be separated by a certain limited distance in configuration space, the energy cannot be made arbitrarily small and therefore will not fall below a certain minimum value. Thus at sufficiently low temperatures the probability of exciting such states is very low and the only excitations will be phonons.

Feynman now deduces an explicit form for the wave function ψ_r and hence determines the energy spectrum of the other excitations. As defined above, ψ_r has a maximum value for configuration A and a minimum value for configuration B, and for low energy states it must vary smoothly in going from A to B. To ensure this, the amplitude of ψ_r is taken to be the number of atoms on sites of A-type configurations minus the number on B configuration sites. This number is $\Sigma f(\mathbf{R})$ where $f(\mathbf{R})$ is a function which is $+1$ if $\mathbf{R}$ defines an A configuration site and -1 if $\mathbf{R}$ defines a B position, and which varies smoothly between these limits. Combining this with the condition that atoms must not overlap, the wave function is written

$$\psi_r = \sum_i f(\mathbf{R}_i)\phi$$

where ϕ, the wave function of the ground state, is introduced to ensure that the atoms do not overlap and to take account of correlations between the atoms. Although Feynman does not specify the motion of the atoms more precisely than by this last equation, he points out that his wave function is consistent with three quite probable modes of motion, viz. the rotation of a small ring of atoms, the excitation of single atoms in a cage, and single atoms vibrating with wave number about $2\pi/a$ where a is the atomic spacing, the surrounding atoms moving about to get out of their path. Feynman then applies the variational principle to find the form of $f(\mathbf{R})$ which gives the lowest energy and thus obtains

$$f(\mathbf{R}) = \exp(i\mathbf{k}.\mathbf{R})$$

and

$$\epsilon(k) = \hbar^2 k^2/2m\,S(k) \qquad \ldots\ldots(4)$$

where S is the Fourier transform of the correlation function

$$S(\mathbf{k}) = \int p(\mathbf{R}) \exp(i\mathbf{k}.\mathbf{R})\, d^3\mathbf{R}.$$

The correlation function, or liquid structure factor, may be deduced either from x-ray data (Reekie and Hutchison 1953) or from neutron diffraction data (Henshaw and Hurst 1953), and the form of the function is shown in figure 5 (*a*). Figure 5 (*b*) shows the resulting curve for the energy given by equation (4), the upper part being similar to Landau's roton spectrum (equation (3)). The height of the minimum of the curve is somewhat high, corresponding to too large a value of Δ; however, the more exact wave functions of Feynman and Cohen (1956) lead to a value for Δ/k of 11·5°K, in much better agreement with the value of about 9·6°K derived from experimental results. We may mention here that the values of Δ, μ and p_0 are best determined from a consideration of the magnitude and temperature dependence of the specific heat and of the 'normal density', to which we refer later. Due to experimental uncertainties, plus of course the fact that the roton spectrum itself is almost certainly only an approximation, it is difficult to obtain a quite definite answer ; however, reasonable agreement results if one takes $\Delta/k = 9{\cdot}6^{\circ}\mathrm{K}$, $p_0/\hbar = 2{\cdot}0 \times 10^{-8}$ and $\mu = 0{\cdot}77m$, where m is the mass of the helium atom (de Klerk, Hudson and Pellam 1954).

The low energy part of the spectrum corresponds to the relation $\epsilon = pc$ and is just the phonon spectrum discussed previously. At first sight this is somewhat surprising as figure 5 (*b*) then suggests that the energy of an excitation can have any value between zero and Δ. This point is still not entirely clear, but Feynman (1955) remarks that the wave functions deduced by the variational method are in general only appropriate in the region of minimum energy. It happens that his wave function for the rotons also turns out to be an exact wave function for phonons of small wave number, and the method therefore gives the lower end of the spectrum correctly. However, this does not imply that the wave function is good for intermediate values of the energy, and therefore phonons and rotons may be quite distinct modes of motion. Indeed, Feynman and Cohen claim that their latest wave function for rotons corresponds roughly to a vortex ring of such small radius that only one atom can pass through its centre, outside the ring there being a slow drift of atoms returning for another passage through the ring.

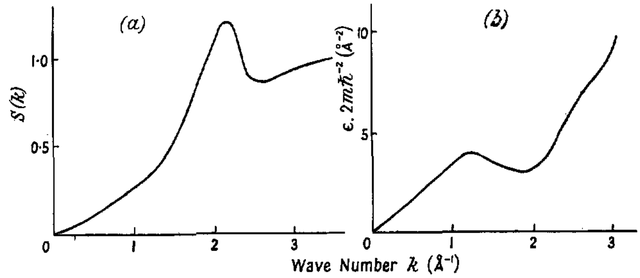

Figure 5. (*a*) The structure factor for liquid helium as determined by neutron diffraction (*after* Henshaw and Hurst 1953) ; (*b*) The energy spectrum for excitations in liquid helium, as calculated by Feynman 1954.

It is to be noted that the above theory does not account, as it presumably should, for the fact that, between about 1°K and the lambda point, the liquid exhibits a negative expansion coefficient. It is possible to relate this coefficient to the three parameters Δ, p_0 and μ, and their dependence on the density of the helium (see, for example, Atkins 1955), but Feynman has not as yet discussed what dependence on density is to be expected. On the other hand, Franchetti (1954) has given an explanation of the negative expansion coefficient in terms of exchange forces ; and Prigogine and his collaborators have given an explanation based on a cell model in which some cells may be doubly occupied and in which the part played by zero-point energy is predominant (Prigogine 1954). The variation of the parameters Δ, p_0 and μ with the density of the helium is also of interest in that it offers, at least in principle, a further test of the theory. Under a pressure of say 25 atmospheres the density is changed by about 10% and we have a liquid to which the theory should still apply but with appreciably different parameters ; thus the lambda point is depressed to about 1·8°K, the specific heat is increased by about 50% (Hercus and Wilks 1955), and the velocities of sound and temperature waves are changed considerably (Atkins and Stasior 1953, Maurer and Herlin 1951).

However, at present some of the data essential for a complete comparison is not available; for example there are as yet no direct determinations under pressure of the 'normal' density (see § 4), nor of the correlation function.

§ 4. THE TWO-FLUID MODEL

We now give the derivation of the remarkable two-fluid model; this derivation was first given by Landau but with several of the intermediate steps omitted, and the position has since been clarified by Dingle, F. London, and particularly by Feynman. Before giving a summary of these arguments, we should perhaps mention that they apply equally well to any form of excitations with given energy and momenta, and do not depend on particular properties of phonons or rotons.

We begin by calculating the total momentum of a mass of helium containing excitations and moving with a constant velocity. To carry out this calculation we need the result that if the energy of an excitation in the liquid at rest is ϵ then the energy that an observer at rest must associate with the same excitation in liquid moving with velocity $\mathbf{v}$ is given by

$$\epsilon_0 = \epsilon + \mathbf{p}.\mathbf{v}. \qquad \ldots\ldots(5)$$

This important result was first deduced by Landau from a consideration of a Galilean transformation in classical mechanics; we give here a rather similar but clearer derivation due to Dingle (1952 a). For an observer who is moving with the liquid and whose coordinate system is indicated by primes, the wave function of an excitation in the liquid may be written

$$\psi = A \exp\left\{\frac{1}{i\hbar}(\epsilon_0 t' - \mathbf{p}.\mathbf{r}')\right\}.$$

The appropriate wave function for an observer at rest (unprimed coordinates) is obtained by the usual transformation

$$\mathbf{r} = \mathbf{r}' + \mathbf{v}t. \quad (t = t')$$

Hence for the observer at rest

$$\begin{aligned}\psi &= A \exp\left\{\frac{1}{i\hbar}[\epsilon t - \mathbf{p}.(\mathbf{r} - \mathbf{v}t)]\right\}\\ &= A \exp\left\{\frac{1}{i\hbar}[(\epsilon + \mathbf{p}.\mathbf{v})t - \mathbf{p}.\mathbf{r}]\right\}.\end{aligned}$$

Thus we have the required result that if the fluid is moving with velocity $\mathbf{v}$ relative to an observer, the energy of an excitation appropriate to his system of coordinates is $\epsilon + \mathbf{p}.\mathbf{v}$. It is perhaps worth mentioning, in view of the arguments shortly to be used, that this result is based purely on mechanical considerations, as yet statistics have played no part in the discussions.

We now consider moving liquid containing a large number of excitations, and apply statistics to determine the total momentum of the assembly. It is convenient to specify that the excitations have an arbitrary drift velocity $\mathbf{v}_n$ with respect to the observer. To fix our ideas consider a cylinder of liquid moving with velocity $\mathbf{v}_s$ and filling a tube moving with velocity $\mathbf{v}_n$, the excitations being in equilibrium with the walls of the tube. Statistical mechanics gives the equilibrium distribution in a gas which has a uniform drift velocity $\mathbf{v}_n$ relative to an observer as

$$n(\epsilon) = [\exp\beta(\epsilon_0 - \mathbf{p}.\mathbf{v}_n) - 1]^{-1}.$$

Treating the excitations as gas-like particles and substituting for ϵ_0 from equation (5) we obtain

$$n(\epsilon) = [\exp\beta\{(\epsilon + \mathbf{p}.\mathbf{v}_s) - \mathbf{p}.\mathbf{v}_n\} - 1]^{-1}.$$

The net momentum associated with the excitations is not zero, as it would be if the liquid were at rest, and is given by

$$\bar{\mathbf{p}} = \int \mathbf{p}\{\exp\beta[\epsilon(\mathbf{p}) + \mathbf{p}.\mathbf{v}_s - \mathbf{p}.\mathbf{v}_n] - 1\}^{-1}(2\pi\hbar)^{-3}\,d\mathbf{p}^3.$$

Making the valid assumption that $\mathbf{v}_s$ and $\mathbf{v}_n$ are small, and expanding to the first order in $\mathbf{v}_s - \mathbf{v}_n$ we find

$$\bar{\mathbf{p}} = -\rho_n(\mathbf{v}_s - \mathbf{v}_n)$$

where ρ_n, an essentially positive quantity with the dimensions of density, is defined by

$$\rho_n = -\tfrac{1}{3}\beta\int p^2[\exp\beta\epsilon(p) - 1]^{-2}\exp\beta\epsilon(p)\,(2\pi\hbar)^{-3}\,d^3\mathbf{p}. \qquad \ldots\ldots(6)$$

Note that in this linear approximation ρ_n will be a function of temperature but not of $\mathbf{v}_n$ or $\mathbf{v}_s$. The total momentum $\mathbf{j}$ associated with unit volume of moving liquid will be the ordinary $m\mathbf{v}$ term plus the momentum associated with the excitations. Thus

$$\mathbf{j} = \rho\mathbf{v}_s + \bar{\mathbf{p}} = \rho\mathbf{v}_s - \rho_n(\mathbf{v}_s - \mathbf{v}_n).$$

Defining a new parameter ρ_s by

$$\rho_s + \rho_n = \rho \qquad \ldots\ldots(7)$$

we have

$$\mathbf{j} = \rho_s\mathbf{v}_s + \rho_n\mathbf{v}_n. \qquad \ldots\ldots(8)$$

The last equation implies that the total momentum of the moving fluid may be formally resolved into two components, one associated with the excitations and the other with the remainder of the fluid. Clearly from the statistical manner in which the quantity ρ_n is derived it is not to be associated with the mass of particular atoms in the system. (This is underlined by the fact that below 0·6°K the values of the specific heat and its T^3 dependence on the temperature indicate that *all* the atoms have vibrational energy—at least in the classical sense—whereas ρ_n turns out to have a value which is only 10^{-4} or less of the density of the liquid.) Nevertheless the terms $\rho_s\mathbf{v}_s$ and $\rho_n\mathbf{v}_n$ are the momenta associated with the motion of the 'background' and of the excitations respectively and thus ρ_s and ρ_n play a role analogous to that of ordinary inertial mass. This analogy finds further support when one calculates the internal energy of a fluid containing excitations; it is found that the internal energy of fluid for which $\mathbf{v}_s$ and $\mathbf{v}_n$ have finite values is greater than that for fluid with zero values of $\mathbf{v}_s$ and $\mathbf{v}_n$, by an amount $\frac{1}{2}\rho_s v_s^2 + \frac{1}{2}\rho_n v_n^2$. Thus ρ_s and ρ_n again behave as inertial masses and this behaviour is the basis of the so-called two-fluid model of helium, in which the motion of the liquid is formally divided into two parts: that of the 'normal fluid', which has 'density' ρ_n and velocity $\mathbf{v}_n$, and that of the 'superfluid', which has 'density' ρ_s and velocity $\mathbf{v}_s$. We again emphasize that the argument leading to equation (6) is a purely statistical one which considers quantum excitations of the *whole liquid*, and that nowhere is there any division of the helium itself into two real parts. Thus the name ' two-fluid model ' is to some extent misleading, as the ' fluids ' are only mathematical abstractions, but it seems to be here to stay. (This model has also been discussed by Dingle 1952 a, H. A. Kramers 1952, and Kronig 1953.)

We now proceed to derive two important equations which describe the motion of the two ' fluids '. The entropy of the helium arises solely from the thermal disorder and therefore will be associated only with the excitations. Transport of entropy will take place at the mean group velocity of the excitations and this may be shown to equal $\mathbf{v}_n$ (Feynman 1954), that is the entropy may be considered as moving with the normal fluid. Thus we can write down the following equation for the conservation of entropy :

$$\frac{\partial}{\partial t}(\rho S)+\operatorname{div}(\rho S\mathbf{v}_n) = 0, \qquad \ldots\ldots(9)$$

S being the entropy of one gramme of liquid helium. This equation also implies that the motion of the fluids is reversible ; this turns out to be a good approximation although irreversible effects have to be considered later.

Finally we need an equation for the forces acting on the superfluid alone, and therefore write down the usual differential expression for the internal energy of a mass of liquid helium

$$dU = TdS - pdV + GdM$$

where G is the Gibbs energy of one gramme of liquid and dM is a change in the mass of the assembly. Let us now increase the mass of the assembly at constant volume by introducing particles which contribute only to the ground state, that is to the superfluid, so that $dS = 0$. As $dV = 0$, we may write $dU = GdM$; the sole result of the operation has been to increase the mass of superfluid by dM, therefore the potential energy of unit mass of superfluid must be G. The equation of motion of the superfluid is thus

$$\frac{\partial \mathbf{v}_s}{\partial t} = -\operatorname{grad} G. \qquad \ldots\ldots(10)$$

Collecting together equations (9) and (10), and adding the usual hydrodynamical equations for $\mathbf{j}$, the total momentum of the liquid, we obtain

$$\frac{\partial \rho}{\partial t}+\operatorname{div}\mathbf{j} = 0 \qquad \ldots\ldots(11.1)$$

$$\frac{d\mathbf{j}}{dt}+\nabla p = 0 \qquad \ldots\ldots(11.2)$$

$$\frac{\partial(\rho S)}{\partial t}+\rho S\operatorname{div}\mathbf{v}_n = 0 \qquad \ldots\ldots(11.3)$$

$$\frac{d\mathbf{v}_s}{dt}+\nabla G = 0. \qquad \ldots\ldots(11.4)$$

Having derived these equations, first given by Landau, we are now in a position to discuss the applications of the two-fluid model.

§ 5. APPLICATIONS OF THE TWO-FLUID MODEL

5.1. *Superfluidity*

Superfluidity was first observed in experiments in which the flow of liquid helium through fine slits and capillaries was seen to take place with virtually a complete absence of friction. Kapitza (1941 a) deduced that if the coefficient of viscosity had a finite value at all, it must be less than 10^{-11} poise, while Allen and Misener (1939) showed that the flow did not obey the ordinary equations of fluid motion. (Actually Allen and Misener's results for slits varying from 10^{-5} to 10^{-3} cm in width were rather complex, but in the narrowest slits the flow appeared independent of the pressure head.) This behaviour is referred to as superfluidity, and can in principle be understood from what has been said already, although some of the detail is still obscure.

Consider the flow of a liquid through a capillary. In general it is necessary to apply a pressure to maintain the flow, and most of the work done by this pressure appears as heat in the liquid. Without going into any details of the viscous processes, we can say that the resistance to flow arises because of a conversion of some of the directed kinetic energy of the liquid into disordered thermal motion. Or again, the relative motion of liquid and capillary sets up additional thermal motion in the liquid, and this is the source of the viscous reaction. Thus to maintain a flow of liquid helium at absolute zero through a tube, work must be done to provide the energy for the creation of any thermal excitations (phonons or rotons) which are produced. However, by a simple argument based on the transformation of coordinates, Landau (1941) showed that excitations can only be formed in this way if the velocity of the fluid through the capillary exceeds a certain critical value

$$\begin{aligned} v_c &= c \simeq 240 \text{ metres sec}^{-1} \text{ (phonons)} \\ v_c{}^2\sqrt{(2\Delta/\mu)} &\simeq 60 \text{ metres sec}^{-1} \text{ (rotons)} \end{aligned} \qquad \ldots\ldots(12)$$

where Δ, μ and c have the same meanings as in equations (1) and (3). Thus if the velocity of the liquid is less than 60 metres sec^{-1} no excitations should be formed, and we reach the surprising conclusion that there will be no viscosity. (Actually it turns out that the values for the 'critical velocities' given in equations (12) are much too high, but we ignore this for a moment.) Hence the liquid at absolute zero should flow through a tube under the action of a vanishingly small pressure gradient at a rate equal to the critical velocity; if it flows more slowly it will be accelerated by the pressure gradient, if more quickly it will be retarded by viscous forces.

The flow of helium at higher temperatures is more complicated. At absolute zero the helium contains no excitations and is entirely superfluid; thus the discussion of the last paragraph shows that the superfluid has no viscosity: hence its name. At higher temperatures, the total motion of the helium must be resolved into that of the two-component (mathematical) fluids. As before, the motion of the superfluid will be free of viscous forces because of the difficulty of producing excitations in it. However, excitations already exist in the normal fluid and these, by interactions with the walls of the tube, can take up energy without restriction. Thus viscous effects should be associated with the motion of the normal fluid, but will be difficult to observe in flow experiments because they will be masked by the great volume of superfluid flow.

There appears to be little doubt that it is the difficulty of creating thermal excitations in the liquid which is responsible for the appearance of superfluidity. However, the mechanism whereby they are excited is obviously not that discussed by Landau, because the value of the critical velocity is found to be not of the order of 60 m $\sec^{-1}$ but only of the order of a few cm $\sec^{-1}$, and is also a complicated function of temperature and slit width. This particular problem is one of the least understood aspects of liquid helium and is discussed later, in the section on turbulence. For the present we confine our attention to the region where there appears to be true superfluidity, that is where the velocities are sufficiently low not to produce excitations in the superfluid.

5.2. *The Normal Viscosity and Andronikashvili's Experiment*

As we have just discussed, measurements of the rate of flow of helium through a capillary do not enable us to determine the viscosity associated with the normal fluid. This ' normal viscosity ' η_n is best found by measuring the viscous force on a body moving through the liquid; in an experiment of this type the superfluid exerts no force on the moving body and the total reaction is that due to the normal viscosity alone. The most straightforward method uses a rotation viscometer and the results of detailed measurements by Heikkila and Hollis-Hallett (1955) are shown in figure 6. We defer a discussion of the form of these results to § 6.1, only noticing here that the viscosity is of the same order as that of a gas.

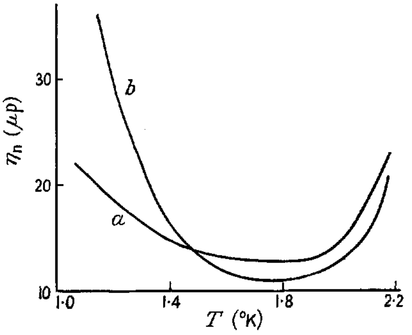

Figure 6. The ' normal ' viscosity of liquid helium as determined (*a*) directly with a rotation viscometer, and (*b*) with an oscillating-disc viscometer plus a determination of the normal density (*after* Heikkila and Hollis-Hallett 1955, and Hollis-Hallett 1952).

One would suppose that the same value for the viscosity would be obtained by making measurements with an oscillating-disc viscometer, but the damping on such a disc is found to decrease steadily as the temperature falls, as is shown in figure 7 (Keesom and MacWood 1938). Before discussing this apparent discrepancy it is necessary to consider the important experiment of Andronikashvili (1948). This author mounted fifty thin metal discs very close together on a common axis, oscillated them about a vertical axis in a bath of liquid helium, and found that below the lambda point the period of the oscillating system decreased with falling temperature. This curious result is accounted for by the fact that while the superfluid is not affected by the motion of the discs, the normal viscosity is sufficient to drag round with the discs all the normal fluid in the narrow spaces between them. Thus the rotating system consists of the pile of discs plus the normal fluid accompanying them ; as the density of the normal fluid varies with temperature, so will the moment of inertia and period of the system. Thus from the experimental observations one can associate a density with the normal fluid, and from what was said in § 4 it would seem that the density so determined should be the density of the normal fluid ρ_n. In fact the values of ρ_n found in this way (figure 8) are in good agreement with those calculated from equation (6) and also with values derived from the velocity of temperature waves which are discussed in § 5.4.

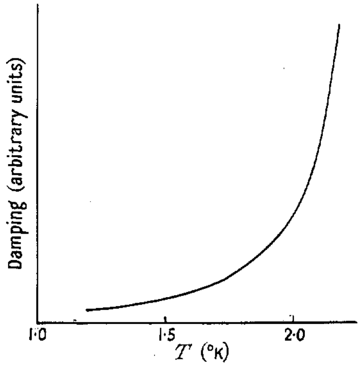

Figure 7. The damping on a disc oscillating in liquid helium, in arbitrary units as a function of temperature (*after* Keesom and MacWood 1938).

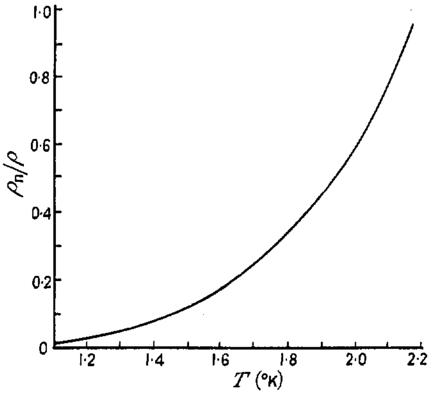

Figure 8. The ' normal ' density of liquid helium ρ_n (*after* Peshkov 1946).

It can now be seen why the damping of an oscillating disc decreases steadily with falling temperature whereas the viscosity does not. The damping of the disc is essentially due to viscous waves which the disc propagates in the liquid, and this damping is proportional to the product of the density and viscosity of the liquid. While for all other liquids the density is constant and the damping of the disc proportional to the viscosity alone, this is not so for helium. Clearly the density in question is not the total density of the helium, as the superfluid is not affected by the motion of the disc, and the position has been analysed in detail by Landau (1944). He showed that the motion of the helium is separable into two independent parts, one part corresponding to a potential flow of superfluid, while the other concerns only the normal fluid and is similar to the solution of the Navier–Stokes equation for a liquid of density ρ_n and viscosity η_n. Thus, in the absence of pressure and temperature gradients, the equation of motion of the normal fluid may be written :

$$\rho_n \frac{d\mathbf{v}_n}{dt} = \eta_n \nabla^2 \mathbf{v}_n.$$

Hence one ultimately obtains values for the viscosity which are in tolerable agreement with those obtained from the rotation viscometer, although, as is seen in figure 6, there is some discrepancy at temperatures below 1·4°K where the position is still somewhat obscure. It may be that the higher values for the normal viscosity given by the oscillation method are due to experimental errors which become more serious at low temperatures, and one possible source of error has recently been reported by Kolm and Herlin (1956). However, it has also been suggested by Ginsburg (1955) that the apparent viscosity is actually greater in the case of the oscillatory motion, because the non-equilibrium conditions are responsible for some form of turbulence in the liquid, as discussed in § 7. Further experiments seem to be required.

5.3. *The Fountain and Mechano-caloric Effects*

The fountain effect is observed when two helium baths at somewhat different temperatures T and $T+\Delta T$ are connected by a very fine capillary, as in figure 9. The capillary acts as a semi-permeable membrane, through which the superfluid may pass but not the normal component ; thus an osmotic pressure Δp is developed corresponding to the excess pressure of normal fluid in the hotter reservoir. This effect was first observed by Allen and Jones (1938), who used a slightly different experimental arrangement and produced such big temperature differences that the pressure flung the liquid up as a jet, hence the name fountain effect. The magnitude of the effect may be derived most simply by noting that in equilibrium there will be no acceleration of the fluids and therefore, by equation (11.4), grad $G = 0$. Using the

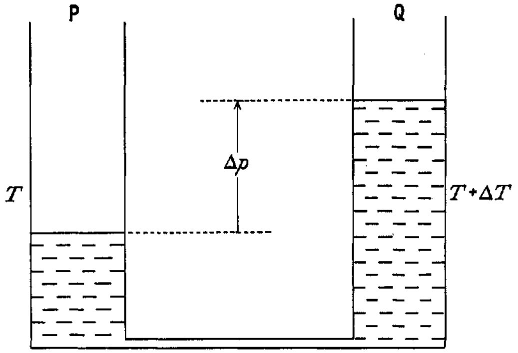

Figure 9. The fountain effect. The vessels P and Q are joined by a fine capillary ; if the liquid helium in Q is hotter than that in P an osmotic pressure Δp is developed between them.

general thermodynamic relation $dG = Vdp - SdT$, one thus obtains the relation

$$\frac{\Delta p}{\Delta T} = \rho S \qquad \ldots\ldots(13)$$

first given by H. London (1939) using a slightly different argument. Alternatively it is possible to calculate the pressure in terms of the constants of the excitations (see, for example, Ward and Wilks 1952). Measurements of the fountain effect are somewhat difficult to carry out, as large differences in the levels are produced by only very small temperature differences. There have been several sets of measurements between about 1°K and the lambda point (for example, Kapitza 1941 a, Meyer and Mellink 1947), and although there is considerable scatter on the results, it seems that H. London's relation is obeyed at least to within the accuracy of the measurements.

We now describe the inverse or mechano-caloric effect (Daunt and Mendelssohn 1939 a) by considering an isolated vessel with a very fine slit at the bottom. Superfluid will flow out through the slit, and as it has no thermal energy it carries away no entropy ; thus the entropy per unit mass of liquid left in the vessel increases, and the liquid becomes warmer. Alternatively, if the vessel is held in an isothermal bath at temperature T, then a quantity of heat Q, the heat of transport, must be removed from the vessel as unit mass of superfluid flows out. If the process is reversible, as we assume, then Q is equal to TS ; this relation has been confirmed to at least a few per cent, any discrepancies being probably within the experimental error (Chandrasekhar and Mendelssohn 1955, Brewer, Edwards and Mendelssohn 1955). Finally we note that as the superfluid leaving the vessel has zero entropy, one might suppose that here is a method of reaching absolute zero and violating the Third Law of thermodynamics. However, even in the narrowest slits, the flow of superfluid will also be accompanied by a flow of the normal fluid, which will only completely vanish in the limiting case of an infinitely narrow slit ; thus helium leaving the vessel will never be at absolute zero. Nor is the process particularly useful for cooling other substances to very low temperatures because the helium has very little entropy at such temperatures (Simon 1950).

Measurements of the fountain pressure and mechano-caloric effect in the region of 1°K and below are of particular interest in connection with an earlier two-fluid theory due to Tisza. Tisza (1938), following F. London (1938), suggested that liquid helium behaved in a manner characteristic of a degenerate Bose–Einstein gas ; using Landau's terminology he seems to have thought of the normal and superfluids as consisting not of *excitations* and background but of *atoms* or groups of atoms in the excited and ground states respectively. Using this picture he was able to account qualitatively for the effects already discussed and also to predict the existence of temperature waves (§ 5.5). Following naturally from this viewpoint, he supposed that the atoms in the superfluid ground state took part in normal Debye type lattice vibrations ; in this case the term S in equation (13) does not refer to the whole entropy of the liquid but only to that part in excess of the entropy of the lattice vibrations (H. London 1939). In other words, on Tisza's model the phonons do not contribute to the fountain pressure. At 1°K the phonons are responsible for about 50% of the total entropy of the helium, at 0·8°K for about 80%, and at 0·6°K for almost 100% ; thus it is possible to differentiate sharply between the theories of Landau and Tisza. Peshkov (1954) has made measurements down to 0·8°K and Bots and Gorter (1956) down to 0·2°K, and it appears that at these temperatures the fountain pressure is given fairly closely by equation (13) with S equal to the total entropy of the liquid. Thus these results, like those concerning temperature waves to be discussed later, confirm Landau's approach in which the normal fluid consists of *excitations* rather than excited *atoms*.

5.4. *Heat Transport*

Transport of heat takes place so readily in liquid helium that impracticably large heat inputs would have to be used to maintain measurable temperature gradients in the bulk liquid, and therefore nearly all the work so far has been done on helium in narrow slits or capillaries. In general the heat flow is not proportional to the cross-sectional area of the liquid; in the narrower channels and for not too large temperature gradients it does vary as dT/dx, but in wider channels it is approximately proportional to $(dT/dx)^{1/3}$. At present we consider the region where the flow depends linearly on the temperature gradient and where the magnitude of the flow may in fact be deduced from Landau's model; we return to the question of the heat flow in wider channels in § 7.1.

Consider two reservoirs of helium at temperatures T and $T+\Delta T$, connected by a very fine slit as in figure 9, so that virtually the whole fountain pressure $\Delta p = \rho S \Delta T$ is developed across it. As the slit is not infinitely narrow (as we assumed when considering the fountain effect) there will be some flow of normal fluid through it, under the action of the osmotic pressure difference Δp, together with a frictionless return flow of superfluid to conserve mass. The magnitude of these two counter current flows will be limited only by the viscosity of the normal component, the velocity of the normal fluid being given by Poiseuille's formula

$$v_{\mathrm{n}} = \frac{B}{\eta_{\mathrm{n}}}\frac{\Delta p}{\Delta x} \qquad \ldots\ldots(14)$$

where Δx is the length of the slit, and B is a constant depending on its geometry. Assuming as before that the relative motion of the two fluids takes place reversibly, the heat flow per unit area, ω, due to the transport of entropy by the normal fluid, will be given by

$$\omega = \rho S T v_{\mathrm{n}}. \qquad \ldots\ldots(15)$$

Substituting (14) in (15) together with London's expression for Δp, we get

$$\omega = \frac{B(\rho TS)^2}{\eta_{\mathrm{n}}}\frac{\Delta T}{\Delta x}. \qquad \ldots\ldots(16)$$

This result has been given both by Andronikashvili (1949) and by F. London and Zilsel (1948), who introduced the name 'internal convection' to describe this mechanism of heat transfer.

As mentioned above, this linear relation between the heat flow and the temperature gradient is only true for not too great heat flows, and in narrow slits of width less than about 100μ (see, for example, figure 10). As the constant B contains the width of the slit to the third power, the experimental verification of equation (16) is by no means easy. Hence it is not surprising that the agreement with some of the earlier measurements is not too close; however, recent work by Winkel, Broese van Groenou and Gorter (1955) seems to confirm that this relation is valid over quite a large region. They show that at spot temperatures the heat flow is proportional to the cube of the slit width as demanded by Poiseuille's equation (figure 11), and that values may be deduced for the viscosity of the normal fluid which are in reasonable agreement with more direct measurements, bearing in mind the accuracy of the experiments.

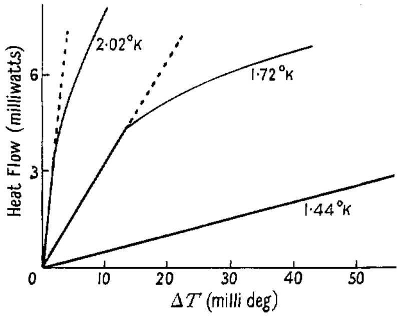

Figure 10. The heat flow through liquid helium in a slit of width 2·4μ as a function of the temperature difference across the slit. The relation is linear only for small heat flows at not too high temperatures (*after* Winkel, Groenou and Gorter 1955).

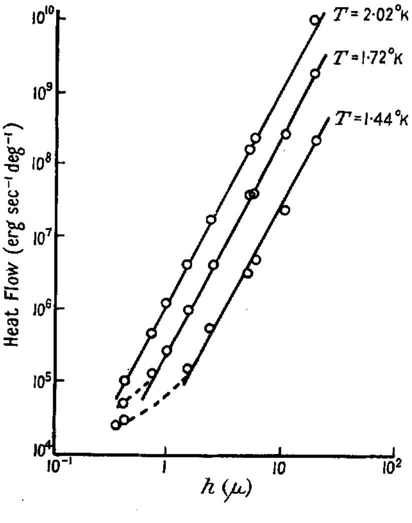

Figure 11. The heat flow through liquid helium as a function of slit width h. The flow varies approximately as the cube of the width in agreement with equation (16) (*after* Winkel, Groenou and Gorter 1955).

5.5. *Temperature Waves*

Equations (11.1) and (11.2) readily lead to the relation

$$\frac{\partial^2 p}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial x^2}$$

the usual equation for the propagation of density or pressure changes in a liquid.

However, there is the additional possibility, with the two-fluid model, of a fluctuation in the relative densities of the normal and superfluids while the total density remains constant ($\mathbf{j} = 0$). As a change in the relative densities of normal and superfluids is equivalent to a change in temperature, an oscillation of the two fluids relative to each other corresponds to an entirely new phenomenon, a true oscillation of temperature. Remembering that the thermal expansion coefficient of helium is so small that the adiabatic and isothermal compressibilities are nearly equal, one can solve the four equations of the model (11.1 to 11.4) and obtain the ordinary sound equation as before plus a second relation (Landau 1941)

$$\frac{\partial^2 S}{\partial t^2} = \frac{\rho_n}{\rho_s}\frac{C}{TS^2}\cdot\frac{\partial^2 S}{\partial x^2}$$

C being the specific heat of unit mass of helium. This is a wave equation for the propagation of variations of entropy or, what amounts to the same thing, of temperature variations ; thus the model predicts the existence of harmonic temperature waves with velocity

$$c_2 = \left(\frac{\rho_s}{\rho_n}\frac{TS^2}{C}\right)^{1/2}.$$

This expression for the velocity, first given by Landau, was confirmed by Peshkov (1946), who excited waves in the liquid by periodic heating ; he detected their presence with a thermometer and was able to tune them to resonance by setting up standing waves. Other measurements were subsequently performed (e.g. Pellam 1949) which showed that, in accordance with the wave equation, pulses of heat propagate through the liquid as compact pulses. Both types of experiment give the same values for the wave velocity, and these (figure 12) together with the known values of T, S and C lead to values of ρ_n/ρ_s which are in good agreement with those determined by other methods.

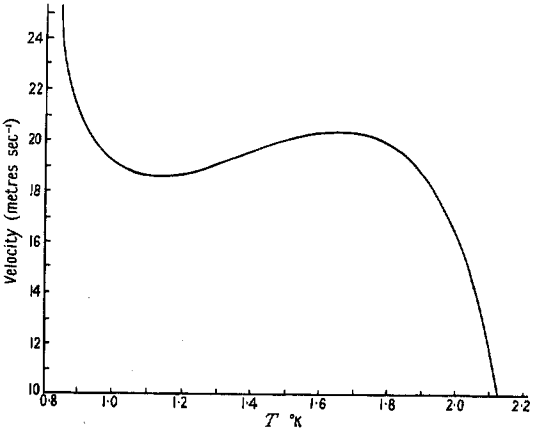

Figure 12. The velocity of temperature waves in liquid helium as a function of temperature (*after* Maurer and Herlin 1951).

Historically, it should be mentioned that the existence of temperature waves was first predicted by Tisza (1938) on the basis of a somewhat different model. However, at the time of writing his first paper, Landau was apparently unaware of this work, and seems to have thought that the velocity c_2 would be associated with sound of the usual type but with another velocity—hence he gave it the name 'second sound'. It was only after a failure (see Peshkov 1946) to excite such waves with a piezoelectric crystal that Lifshitz (1944) showed that a much more effective technique would be to generate a periodically varying temperature.

At temperatures appreciably below 1°K the position becomes more complex and, as is discussed in § 6.3, it now appears that in general a wave-like propagation of temperature variation is not possible below about 0·7°K. However, it is important to note that resonance techniques (Peshkov 1952) as well as pulse measurements confirm the existence of temperature waves down to at least 0·85°K, and that the velocity rises rapidly as the temperature falls, in agreement with Landau's formula. In contrast, Tisza's theory, which is based on fundamentally different concepts (§ 5.3), predicts that the velocity should fall in this region.

5.6. *The Momentum of Heat Flow*

As discussed in the last sections, heat flow in liquid helium is accompanied by motion of the two fluids; therefore as the fluids have momentum we arrive at the unusual result that a flow of heat has momentum associated with it. This momentum was first observed by Kapitza (1941 b), who showed that a flow of heat impinging on a vane mounted on a torsion wire gave rise to a pressure which deflected the vane. However, Kapitza's experiment did not show the dependence of the pressure on temperature which one would now calculate, probably because of the rather uncertain geometry of his apparatus. The momentum flux associated with each fluid is $\rho\mathbf{v}.\mathbf{v}$; thus in a linear flow of heat there is a reaction pressure $\rho_s v_s^2 + \rho_n v_n^2$ on the source, due to the inertia of the fluids. Both fluids give positive contributions, as positive momentum is associated with positive velocity, and negative momentum is associated with negative velocity. Using the equations of the two-fluid theory one readily obtains the relation

$$\rho_s v_s^2 + \rho_n v_n^2 = \frac{\omega^2}{c_2^2 \rho C T}. \qquad \ldots\ldots(17)$$

Hall (1954) has measured the reaction pressure on heating elements immersed in liquid helium and shown that it varies as the square of the heat flow, and as temperature in the rather complicated way predicted by the above expression. The form of this temperature dependence is shown in figure 13, together with experimental points arbitrarily normalized to allow for edge effects associated with the finite size of the heaters.

Another ingenious demonstration of the existence of the relative motion of the two fluids in a temperature wave has been given by Pellam and Morse (1950) and Pellam and Hanson (1952). As the experiment involves the use of a modified Rayleigh disc in liquid helium, we briefly recall that, in general, a disc in a moving fluid tends to align itself crosswise to the direction of the particle velocity. The torque acting on the disc is proportional to the kinetic energy of the motion, that is to ρv^2, where ρ is the density of the fluid and v the particle velocity. Because the magnitude of the torque is quadratic in the particle velocity, there will be a resultant mean torque on the disc even though the velocity varies harmonically and is directed in opposite directions for successive half-cycles. Now the motion of the two fluids in a temperature wave in liquid helium has kinetic energy $\frac{1}{2}\rho_n v_n^2$ associated with the normal fluid and $\frac{1}{2}\rho_s v_s^2$ associated with the superfluid. Both these terms give rise to torques which will be additive, even though it follows from equation (8) that the particle velocities in the two fluids must be in antiphase if we assume that there are no density fluctuations ($\mathbf{j} = 0$). The total torque may be readily related to the heat current in the temperature wave, and calculated values for a given disc are shown in figure 14; the experimental results of Pellam and Hanson (1952) are in satisfactory agreement.

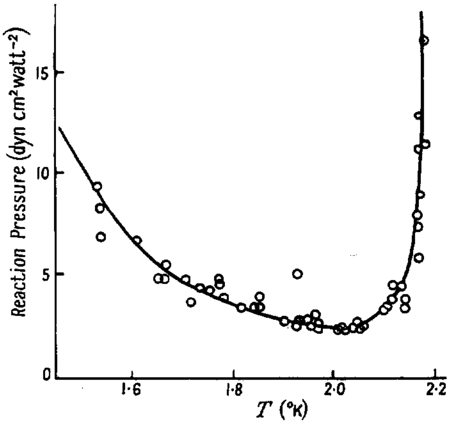

Figure 13. The momentum of heat flow exemplified by the reaction pressure on a source of heat in liquid helium. The full curve gives the theoretical expression of equation (17), and the points the normalized experimental values (*after* Hall 1954).

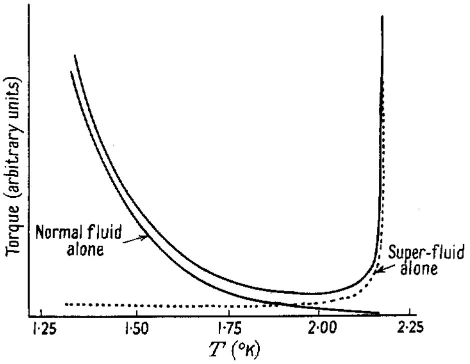

Figure 14. The torque on a Rayleigh disc in liquid helium due to temperature waves, as a function of temperature. The upper curve shows the total torque ; the lower curves give the torques associated with the motion of the normal and superfluids respectively (*after* Pellam and Hanson 1952).

§ 6. THE EXCITATION GAS

In the preceding applications of the two-fluid model the normal fluid has been regarded as an abstract continuous medium, but many of the results could have been obtained by considering the fluid as a gas-like assembly of excitations. Thus the fountain pressure is the excess pressure of the excitation gas in the hotter vessel, and a flow of heat down a temperature gradient corresponds to a flow of excitations maintained by the fountain pressure. In this section we consider more complex parameters of the liquid, such as the coefficients of viscosity and sound absorption, whose magnitudes can only be derived from a consideration of the excitations themselves. Both these coefficients characterize processes in which the distribution function of the excitations is disturbed from its equilibrium value, so that a flow of energy is set up tending to restore the equilibrium. Such behaviour has been examined in a very general way by Khalatnikov (1952 a), who considers the most general form of excitation defined solely by its energy ϵ and momentum p. Khalatnikov takes the four linear equations of motion (11.1) to (11.4) together with the Landau expression for the Hamiltonian of an excitation ($H = \epsilon(\mathbf{p}) + \mathbf{p}.\mathbf{v}_s$) and then calculates the flow of energy due to a departure from equilibrium. In general this flow may result from any or all of three possible types of disturbance, and Khalatnikov associates with each type a characteristic ' kinetic coefficient ' α, β or γ. α characterizes the energy transfer associated with the 'normal' viscosity and β the transfer associated with the second viscosity. (This second viscosity is sometimes also described as the dynamic viscosity; it is associated with density changes in a system, as is described, for example, by Karim and Rosenhead (1952).) The third coefficient γ is the ' thermal coefficient ' and concerns the transfer of energy when there is a temperature gradient in the excitation gas, but although it is somewhat analogous to the thermal conductivity of an ordinary gas the analogy has to be used cautiously. Thus γ has no immediate relation to the flow of heat in liquid helium itself, which as we have seen is brought about by the relative and *reversible* motion of the two fluids under the influence of the fountain pressure. The thermal coefficient concerns a second-order effect depending on the fact that if a temperature gradient is established in liquid helium there will also be a concentration gradient in the normal fluid. To a first approximation this will not affect the value of the heat flow which is given by equation (16); however, it does give rise to a small flow of heat by an *irreversible mechanism* which is of consequence when one considers the attenuation of temperature waves.

6.1. *The Viscosity of the Normal Fluid*

The viscosity of liquid helium is that of the normal fluid, that is of the excitation gas, and the coefficient of viscosity η_n has been calculated on this basis by Landau and Khalatnikov (1949) using standard kinetic theory methods. Below about 1·8°K, where the excitation gas is not too dense and the interactions not too strong, the excitations are assigned scattering cross sections and mean free paths; then by setting up the transport equation an expression is obtained for the momentum transfer and hence the viscosity. The calculations are somewhat lengthy because collisions may occur between phonons and phonons, phonons and rotons, and rotons and rotons; also the results for each type of collision must be averaged over the whole of momentum space. It is not possible here to go into any of the details and we merely indicate the outline of the treatment and the results obtained.

As both the phonons and the rotons transport momentum, there is both a phonon and a roton contribution to the viscosity. For lack of more detailed information a roton is treated as a sort of particle; interactions between rotons lead to a viscosity which is independent of temperature but of indeterminate magnitude. (This independence of temperature is quite analogous to the fact that the viscosity of a perfect gas is independent of its density.) In the case of the phonons it is possible to obtain an exact expression for the viscosity, as their interaction energies are related to the curvature of the phonon part of the excitation spectrum. (As may be seen from figure 5 (*b*), this spectrum is only given by the linear relation $\epsilon = pc$ for values of p close to zero.) The phonon viscosity is limited principally by the scattering of the phonons by the heavier rotons, and therefore increases with falling temperature because of the rapidly decreasing number of rotons. The calculated value is in quite good agreement with the observed rise in the normal viscosity at temperatures below 1·4°K. However, the close numerical agreement which is obtained is probably somewhat fortuitous because many of the required data (e.g. $\partial\Delta/\partial p$, $\partial^2\Delta/\partial p^2$) have to be deduced or estimated from other experiments rather indirectly, as for example is done by Atkins (1955) ; Landau and Khalatnikov themselves remark that no great accuracy can be expected. Also there is still some uncertainty over the experimental value of the viscosity at low temperatures as mentioned in § 5.1. However, the theory gives a good semi-quantitative account of the behaviour of the viscosity.

6.2. *The Attenuation of Pressure and Temperature Waves*

The coefficient of the absorption of sound or pressure waves for a frequency of 14 Mc/s is shown in figure 15. Above about 3°K the absorption is accounted for by the classical expression for the attenuation due to viscosity and thermal conduction ; the high value between 2° and 3°K is associated with the lambda transition, and has been discussed, in detail, by Pippard (1951). For the present, however, we are concerned with the region below the lambda point where the behaviour is obviously quite different from that above, and where the absorption has been calculated by Khalatnikov (1950, 1952 b) on the basis of the excitation gas model. The passage of a pressure wave disturbs the equilibrium of the helium, and therefore of the excitation gas ; then, as is well known, there are absorption effects except in the two limiting cases when equilibrium is restored either very quickly or very slowly with respect to the period of the sound waves. The absorption arises by virtue of the transfer of energy in non-equilibrium conditions

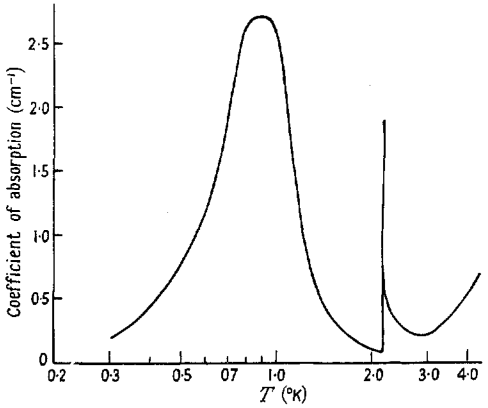

Figure 15. The absorption of 14 Mc/s sound waves in liquid helium as a function of temperature (*after* Pellam and Squire 1947, Chase 1953, Newell and Wilks 1956).

by the three mechanisms discussed in § 6; by solving the appropriate equations for the wave motion Khalatnikov obtains expressions for the attenuation of sound waves in terms of the coefficients of first and second viscosity and of the thermal coefficient.

In his first paper Khalatnikov (1950) pointed out that the viscosity of the normal fluid can only account for about one-fifth of the attenuation observed by Pellam and Squire (1947). As the coefficient of expansion of helium is small, any temperature changes due to the passage of the sound wave will also be small and therefore Khalatnikov postulated that the excess absorption was due to second viscosity rather than to the thermal coefficient. The calculation of the magnitude of the second viscosity requires a knowledge of two relaxation times, characteristic of the approach to equilibrium of an excitation gas in which the number of phonons and the number of rotons have non-equilibrium values. In principle the relaxation times can be calculated, but in practice this is very complicated; therefore Khalatnikov deduced their temperature dependence and then obtained their absolute magnitudes by fitting his expression for the attenuation to the experimental values. Having obtained these two parameters, he was able to predict from Pellam and Squires' results, which only extended down to 1·6°K, that there should be a maximum in the absorption at about 0·9°K and that it should be accompanied by dispersion (Khalatnikov 1950). Quite some time afterwards, both the maximum and the dispersion were observed and found to be approximately of the predicted magnitude (Chase 1953, Chase and Herlin 1955, Newell and Wilks 1956); at the lowest temperatures, however, the absorption is much greater than was expected.† Finally, we should mention that the calculations are very involved and that, as in the calculation of the normal viscosity, some of the parameters are only known approximately; thus too much importance should not be laid on detailed numerical agreement. However, the agreement between the predictions and what was afterwards discovered is striking.

Khalatnikov also considered the attenuation of temperature waves in the bulk liquid. In his first paper (1950) he calculated the attenuation due to first and second viscosity, while in his second paper he took into account the process characterized by the thermal coefficient, the effect of which was to increase the attenuation by a factor of about three. Experimental values of the absorption down to 1°K have been given recently (Atkins and Hart 1954, Pellam and Hanson 1954) and although there is some discrepancy between the two sets of results, it seems that the attenuation is substantially as was predicted, rising as the temperature falls. Here again theory anticipated experiment.

6.3. *Mean Free Path Effects*

Although the normal fluid is an assembly of discrete excitations, equations (11.1) to (11.4) implicitly assume that the two fluids are continuous (mathematical) media. We now consider what modifications to the results already obtained are involved when the structure of the normal fluid is taken into account, and it is useful to begin by considering the propagation of temperature waves. Suppose

---

† It now seems likely that the excess absorption is associated with temperature differences set up by the wave, which may no longer be ignored at the lowest temperatures (Dransfeld, Newell and Wilks, to be published.)

we ignore the superfluid for the moment and consider only the excitation gas, which is the normal fluid. Then it may be shown (Ward and Wilks 1951, 1952, Dingle 1952b) that, provided the conservation laws of energy and momentum are valid for interactions between the excitations, a periodic disturbance of the distribution function will propagate itself as a wave motion. But temperature variations may be regarded as particular kinds of changes in the distribution function, and therefore the excitation gas will propagate temperature waves. Dingle also shows that the velocity of the waves so arrived at is given by $c_2{}^2 = (\rho/\rho_n)\,TS^2/C$ which approximates to the Landau expression at low temperatures, when ρ_n is small and $\rho \simeq \rho_s$.† At higher temperatures the deviations from the Landau formula are considerable because it is no longer permissible to ignore the motion of the superfluid. The point is that fluctuations in the gas of excitations correspond to changes in the normal density ρ_n, and as the ordinary density of the helium is assumed to remain constant, the relation $\rho_n + \rho_s = \rho$ implies a drift of superfluid associated with the temperature fluctuations. At low temperatures where the density fluctuations are small this drift may be neglected, but at higher temperatures it is essential to take it into account as is done in the two-fluid model.

From the above discussion it is clear that an essential criterion for the propagation of second sound is that the mean free path of excitations between interactions should be small compared with the wavelength of the second sound. Now the work of Landau and Khalatnikov, discussed above, shows that the collisions which propagate the second sound are those considered when calculating the viscosity. From their work on the viscosity it is known that the mean free paths between collisions increase very rapidly with falling temperature, so that by about 0·6° they are of the order of 1 cm. This figure is considerably greater than the wavelength of the second sound in any experiments so far performed and one would therefore expect the propagation of temperature waves to break down by this temperature (Gorter 1952, Atkins 1953), as indeed is shown by several experiments (for example, those of Kramers, Peski-Tinbergen, Wiebes, van den Burg and Gorter 1954). Figure 16 (*a*) shows the shape of a temperature pulse received at 0·9°K as viewed on an oscillograph trace triggered by the transmitted pulse ; the pulse is quite sharp and is of about the same width as the transmitted signal. Figure 16 (*b*) shows the corresponding received signal at about 0·2°K, viewed on the same time scale ; the pulse arrives much sooner, as would be expected from the calculated velocity of temperature waves at this temperature, but it has become so very broad that a wave equation of propagation is hardly applicable. As all these experiments

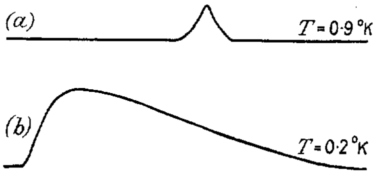

Figure 16. The shape of a heat pulse transmitted through liquid helium and viewed on an oscilloscope trace, as a function of temperature ; at 0·9°K the pulse is quite compact and characteristic of a wave-like form of propagation (*after* Atkins and Osborne 1950).

---

† The values of ρ_s at 1·0 and 1·2°K are about 0·99 ρ and 0·97 ρ respectively, so that even at 1·2°K this formula gives the velocity correct to 1½%.

have been conducted in comparatively narrow cylinders of helium, the mean free paths of the excitations have also been greater than the diameter of the container. Thus the flow of heat in the helium should be very similar to the flow in a dielectric medium whose thermal conductivity is limited only by boundary scattering. Ziman (1954) has worked out in detail the change of pulse shape which would be expected on such a model, and obtains results very similar to those observed by Atkins and Osborne (1950) (figure 16).

Similar mean free path effects are to be expected in the behaviour of the viscosity; thus η_n should fall below the Landau and Khalatnikov value when the mean free paths characteristic of the viscous processes approach the dimensions of the experimental apparatus. This effect should be most marked in narrow slits at low temperatures but very few data are yet available. However, the results of Winkel, Broese van Groenou and Gorter (1955) give an anomalously high value of the heat flow in an $0{\cdot}4\mu$ slit at the lowest temperatures, and it may well be that such effects are limiting the viscosity, the mean free path calculated by Landau and Khalatnikov being of the order of the slit width at this temperature.

§ 7. TURBULENCE

As we mentioned when discussing the derivation of ρ_n, the equations of motion for the liquid are valid only to the first order in $v_s - v_n$, or what amounts to the same thing in v_s or v_n; thus we might expect them to be inadequate for experiments involving high velocities of the fluids. Several authors have discussed the form that the second-order terms should take, and there is very little agreement between them (see, for example, Daunt and Smith 1954). However, the only experiment in which it appears necessary to consider quadratic terms is the propagation of temperature waves of high amplitude, where shock effects are observed (Osborne 1951); either the back or front edge of a pulse becomes steeper according to the temperature (figure 17). Khalatnikov (1952 c) has shown that this behaviour is consistent with equations of motion quadratic in v^2 given by Landau (1941), and there seems little need to investigate this particular question further. It is generally found that before the velocities of flow become sufficiently large to call for the use of these quadratic terms, other processes occur which drastically modify the behaviour of the helium. These processes, which we characterize quite loosely by the name turbulence, are by no means fully understood, so we first present the experimental facts and then go on to mention possible interpretations.

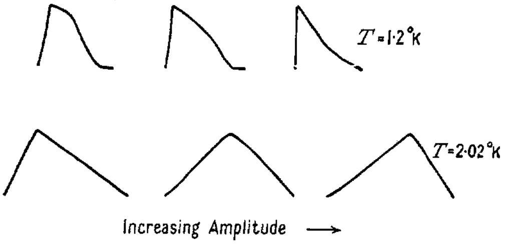

Figure 17. The shape of heat pulses transmitted through liquid helium and viewed on an oscilloscope trace, as a function of pulse amplitude. Below 1·96°K second-order effects, at high amplitudes, result in the front edge becoming steeper; above 1·96°K the back edge becomes steeper (*after* Osborne 1951).

7.1. *Viscosity and Heat Flow Experiments*

Kapitza (1941 b) observed that the heat flow through an annulus of helium between a thin tube and a narrow rod was greatly reduced if the rod was rotated at speed. This was the first observation of the now generally recognized behaviour of the liquid, that when the velocities of the constituent fluids exceed ' critical ' values of the order of a few cm sec^{-1}, its properties differ considerably from those outlined above. The transition to the perturbed state is well marked, as is illustrated by measurements of the viscosity and heat flow. Hollis-Hallett (1950, 1955) has shown that the damping of an oscillating disc rises steeply at a certain amplitude of swing, in a manner which suggests the onset of new dissipative forces entirely absent at lower amplitudes (figure 18). Similarly, Hung, Hunt and Winkel (1952) showed that if the temperature gradient in a fine slit exceeds a certain value, then the heat flow is no longer governed by equation (16), but assumes lower values and even becomes proportional to the cube root of the temperature gradient. Lacking a better name, we will refer to this condition characterized by higher dissipative forces as one of turbulence, although without committing ourselves as to its precise nature.

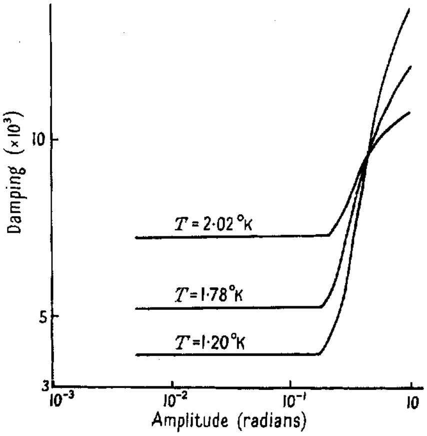

Figure 18. The damping on a disc oscillating in liquid helium as a function of amplitude (*after* Hollis-Hallett 1955).

A valuable step in analysing these experiments was taken by Gorter and Mellink (1949) who postulated that the equations of motion of the two-fluid model would still be valid in the turbulent region provided that additional frictional terms were included. Besides the viscous forces η_n acting on the normal fluid alone, they postulated a mutual friction force acting between the two fluids and of the form $F_{sn} = A\rho_s\rho_n(v_s - v_n)^3$, the cube relation being chosen to give the correct dependence of heat flow on temperature gradient. Hence they write

$$\rho_s\frac{d\mathbf{v}_s}{dt} = -\frac{\rho_s}{\rho}\operatorname{grad}p + \rho_s S \operatorname{grad} T - A\rho_s\rho_n(\mathbf{v}_s - \mathbf{v}_n)^3,$$

$$\rho_n\frac{d\mathbf{v}_n}{dt} = -\frac{\rho_n}{\rho}\operatorname{grad}p - \rho_s S \operatorname{grad} T + A\rho_s\rho_n(\mathbf{v}_s - \mathbf{v}_n)^3 + \eta_n\nabla^2\mathbf{v}_n, \qquad \ldots\ldots(18)$$

these equations being equivalent to Landau's if $A = 0$ (Dingle 1949).

These equations account moderately well for the heat flow experiments, although the values of the constant A deduced from measurements made with different sized slits at different temperatures vary by over a factor of two. However, they do not seem capable of accounting for the flow of the liquid through narrow channels such as described by Hung, Hunt and Winkel (1952) and more particularly by Atkins (1951). As these flow experiments are both difficult to carry out and to interpret, too much weight should not perhaps be attached to some quantitative discrepancy, but even so the equations appear to be inadequate.

Equations (18) as they stand also fail to interpret the oscillating disc experiments of Hollis-Hallett (1952); for example, figure 19 shows the excess damping which would be expected using a value of mutual friction deduced from the heat flow measurements. Actually the excess damping observed varies with temperature in a rather similar manner to the density of the superfluid, suggesting the presence of a frictional force F_s which acts in the superfluid alone. The existence of such a superfluid viscous term in addition to the mutual friction force is confirmed by measurements of the viscosity using a rotation viscometer (Heikkila and Hollis-Hallett 1955). In the steady state the mutual friction term cannot give rise to any torque, but at the higher velocities a contribution to the torque is observed over and above that due to the viscosity of the normal fluid. Finally, to round off the argument, we mention that the flow experiments of Atkins (1952) would seem to indicate that if only a force F_s were present (and no mutual friction), then the heat transport that would be deduced from measurements of fluid flow would be grossly too large. Thus to account for the above results it seems that both terms F_s and F_{sn} must be added to the equations of motion. Of course there is no need to postulate two different mechanisms; if some form of viscosity is set up in the superfluid, one would expect it to be accompanied by some interaction with the normal component.

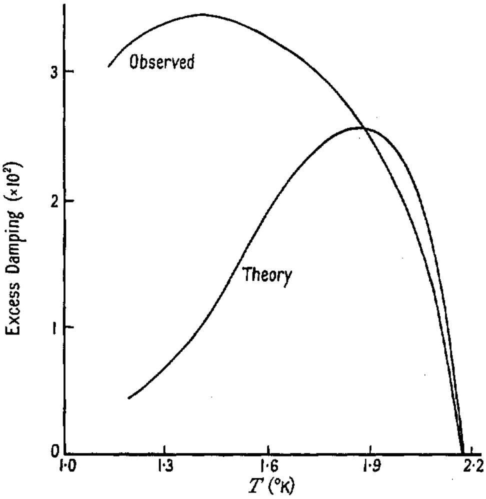

Figure 19. The excess damping on a pile of discs oscillating in liquid helium which results when large amplitudes of oscillation are used. (Amplitude and period of oscillation: 0·215 radian and 2·60 sec respectively.) The curves show the experimental values and the values predicted by the Gorter–Mellink theory (*after* Hollis-Hallett 1952).

7.2. *The Critical Velocity*

An important feature of turbulence is shown in the results of Allen and Misener (1939) for the velocity of flow in slits 10^{-4} to 10^{-5} cm wide (figure 20). The velocity has a value almost independent of pressure but depending on the particular conditions of the experiment; this value is presumably the critical velocity, above which the viscous forces set in so suddenly that any further increase of velocity is energetically impossible. The figure also shows that the rate of flow, and therefore the critical velocity, decreases by a factor of about 4 for an increase in slit width of about 30 times. A dependence of critical velocity on the smallest linear dimension of the liquid is observed in many different types of experiments and appears to be one of the most fundamental aspects of the onset of turbulence. Thus Atkins (1951) has observed the flow of helium through capillaries up to 10^{-2} cm diameter and shown that the critical velocity continues to decrease with increasing diameter up to at least 10^{-2} cm, although in this case the flow of the normal fluid becomes so considerable that it is difficult to obtain accurate values. The experiments with oscillating disc and rotation viscometers, referred to above, give a clear indication of the onset of turbulence; and in the case of the rotation viscometer it seems reasonable to assume that the critical dimension is the gap between the two cylinders.

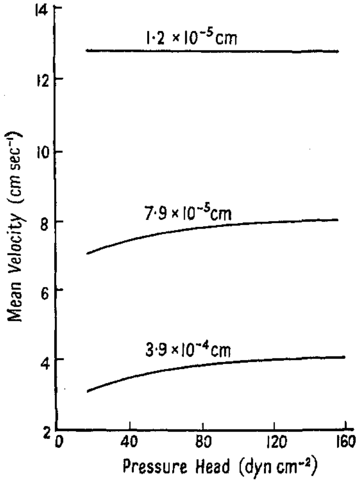

Figure 20. The velocity of flow of liquid helium through fine channels at 1·2°K. Channel width as shown (*after* Allen and Misener 1939 and Atkins 1952).

Several recent experiments have helped to clarify the position by showing how the mutual friction depends on the velocities of the two fluids. Winkel, Delsing and Gorter (1955) have measured the fountain effect across slits of such width that the full fountain pressure is not developed because the slit is not completely impermeable to the normal fluid. From the equations of motion they show that the reduction in fountain pressure in their experiments is directly proportional to the constant of mutual friction, and are therefore able to obtain values for this constant. Winkel (1955) has shown that if helium is caused to flow from vessel P to Q through a narrow slit by the supply of heat to Q, then the motion does not cease immediately on cutting off the supply of heat. Observations of this 'overshoot' effect by Winkel, Delsing and Poll (1955) lead to further values for the mutual friction and also to accurate values of the critical velocity. Finally, Winkel, Broese van Groenou and Gorter (1955) have made and analysed new measurements of heat flows in narrow slits at sufficiently low heat current densities so as

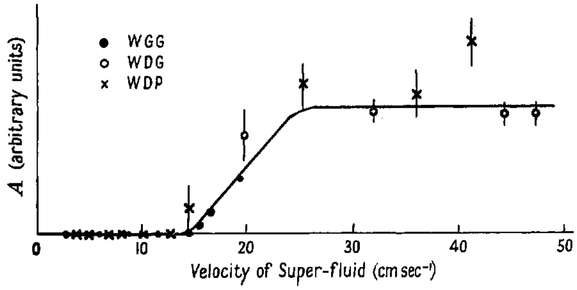

Figure 21. The coefficient of mutual friction A as a function of the velocity of the superfluid v_s (in arbitrary units).
WGG : Winkel, Groenou and Gorter (1955) ;
WDG : Winkel, Delsing and Gorter (1955) ;
WDP : Winkel, Delsing and Poll (1955).
(*After* Winkel, Groenou and Gorter 1955.)

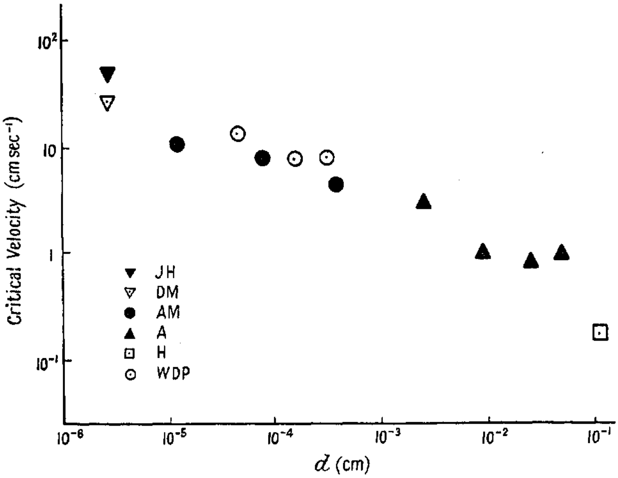

Figure 22. The critical velocity of the superfluid as a function of the smallest linear dimension of the helium d.
JH : from film thickness (Jackson and Henshaw 1950) ;
DM : from film thickness (Daunt and Mendelssohn 1939) ;
AM : from flow (Allen and Misener 1939) ;
A : from flow (Atkins 1951) ;
H : from viscosity (Hollis-Hallett 1952) ;
WDP : from 'overshoot effect' (Winkel, Delsing and Poll 1955).
(*After* Dash 1954 and Winkel, Delsing and Poll 1955.)

to be able to observe the transition to turbulence. From all these data it is possible to decide which is the crucial velocity, that of the normal or of the superfluid or their relative velocity. Figure 21, taken from Winkel, Broese van Groenou and Gorter, shows values of the mutual friction constant A, obtained by the different Leyden methods, for a given temperature and slit width, plotted against the velocity of the superfluid. We see that in all the rather different experiments there seems to be a unique relation between the values of A and the superfluid velocity. Moreover this would not be the case if the normal or relative velocities were taken as abscissae, for in flow type experiments $V_n \simeq V_s$, but in heat transport experiments $V_n \simeq V_s \rho_s / \rho_n$. Thus it appears that it is the velocity of the superfluid which gives the criterion for the onset of turbulence; similar conclusions have also been reached by Dash (1954) and Vinen (1955) using somewhat different arguments. Figure 22 summarizes how the critical velocity of the superfluid depends on the linear dimensions of the helium; it is seen that the results from many different experiments fall into a common pattern.

7.3. *Vortex Lines*

The experimental evidence summarized in the last section suggests that at a certain velocity, of the order of a few cm sec^{-1}, and which depends on the dimensions of the apparatus, the superfluid becomes subject to frictional forces. This implies that some forms of excitation are being created in the superfluid, but, on Landau's theory, excitations can only be formed if the velocity of flow is at least 60 metres sec^{-1} for rotons and 240 metres sec^{-1} for phonons (§ 5.1). Clearly some other mechanism has yet to be taken into account and we now consider what may have been omitted in Landau's approach.

The crux of the matter seems to be that the argument used by Landau to deduce values for the critical velocity implicitly assumes that the flow of the liquid is irrotational potential flow. (It assumes that when an excitation is created the velocity of the whole mass of moving liquid is reduced by the same small amount.) This point was well appreciated by Landau and he had already shown that the flow was potential ($\mathrm{rot}\, v_s = 0$) by arguments based on his quantum hydrodynamics. However, we have previously mentioned objections to the hydrodynamics, and these have recently been confirmed by an experiment which shows that on occasion the flow is definitely not potential. Hall (1955) has measured the torque needed to accelerate a cylindrical vessel containing liquid helium to a constant angular velocity, and has thus shown that when a steady state has been reached the whole mass of the helium is rotating. At first sight this result might seem to be at variance with Andronikashvili's experiment (§ 5.2) in which velocities of the same order of magnitude were involved but only the normal fluid took part in the motion. However, the two experiments are not inconsistent, for in Hall's experiment it took about a minute for the superfluid to be accelerated to a steady velocity, and the acceleration was disproportionately small at the beginning of the motion. Thus one would hardly expect the superfluid to be set in motion at all in Andronikashvili's experiment which concerned oscillatory motion of period about a second. It is this time effect which also accounts for the fact that temperature waves are not affected by turbulence, even though the velocities of the fluids may exceed the critical value for uni-directional flow.

Given time it appears that the whole mass of liquid helium can be brought into rotation, and it now becomes necessary to consider what configuration the rotating helium is likely to adopt. It has been realized for some years (H. London 1946, F. London 1954, Landau and Lifshitz 1955) that the most energetically favourable state would be one in which cylindrical annuli of the liquid performed irrotational rotations about the axis, the annuli being separated by vortex sheets whose separation decreased with increasing angular velocity. Onsager (1949) has postulated that the vortex sheets consist of arrays of vortex lines and that these will be quantized. This concept has been discussed in detail by Feynman (1955) but as yet only in a rather qualitative way. However, although much is yet to be analysed, it seems clear that such vortices could be a source of mutual friction, and, being quantized, would presumably only be created under certain critical conditions.

7.4. *Rotating Helium*

The clearest evidence for the existence of the vortex lines discussed in the last section is an experiment on the propagation of temperature waves, by Hall and Vinen (1955). These authors mounted a temperature wave resonator, complete with heater and thermometer, in a vessel of liquid helium and rotated the whole assembly at constant velocity. When a steady state had been reached, it was found that there was an additional attenuation of the waves which was proportional to the angular velocity of the system. The absorption increased if the cryostat was disturbed by shaking or if the rotating system was accelerated, but the most significant result was the presence of additional absorption just due to steady rotation. A rather similar experiment has also been performed by Wheeler, Blakewood and Lane (1955), who observed a greater attenuation probably because the geometry of their apparatus was such as to induce more turbulence in the liquid.

Thus there is good ground for supposing that vortex lines are the key to the behaviour of liquid helium in the super-critical condition, but several problems have still to be elucidated. There is still no really satisfactory treatment relating the critical velocities discussed in § 7.2 with the dimensions of the apparatus, although Feynman (1955) has made a start on this problem, and shown how the critical velocity could decrease with increasing slit width. There is also no explanation at all as to why the heat flow in the super-critical region varies approximately as the cube root of the temperature gradient. Again it is not entirely clear how the vortices are related to the two-fluid model, although they probably should be considered as another form of excitation with a very large scattering cross section. As the total energy per unit volume of the vortex lines is small compared with the heat energy (Feynman 1955), they will probably make little contribution to the density of the normal fluid. This view seems to be confirmed by two recent experiments on rotating helium which involve the density of the normal fluid: Hall and Vinen (1955) found that the velocity of temperature waves is unaffected by rotation, and Andronikashvili and Kaverkin (1955) have shown that the fountain pressure is also unaffected.

§ 8. THE HELIUM FILM

One of the most characteristic features of helium II is the phenomenon of film flow first described by Rollin (1936) and Rollin and Simon (1939), and investigated in an elegant series of experiments by Daunt and Mendelssohn (1939 b). The essence of the experimental arrangement is shown in figure 23; if the lower part of an empty beaker is lowered into a bath of helium, helium from the bath creeps up over the rim of the beaker until the liquid levels inside and outside are equal.

Similarly, if a beaker containing liquid is placed above the bath, the helium creeps over the rim and returns to the lower level in the bath. To account for this behaviour one recalls that if a solid surface is in contact with any liquid having a positive angle of contact, a thin film of liquid forms on it due to van der Waals forces. In the case of helium this film will be of the order of 10^{-6} cm thick, and below the lambda point the superfluid component of the liquid in the film will be fully mobile, so that mass flow can take place under extremely small potential gradients. (In the beaker experiment this potential is provided by the variation of the free energy of the vapour with height above the lower liquid level.) Although this brief outline sums up the essential behaviour of the film, there is still much uncertainty over the details of the picture, principally because of the difficulties of establishing the experimental facts, and some of these points are now considered in more detail.

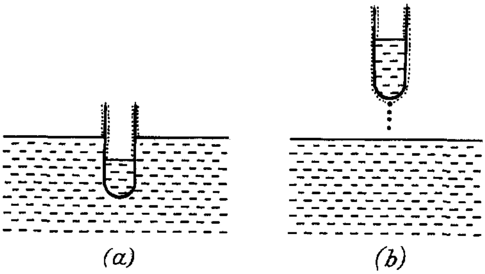

Figure 23. (*a*) If an empty beaker is lowered into a bath of helium II, film flow occurs over the surface of the beaker until the levels are equalized. (*b*) If the beaker is now lifted above the bath, the helium flows out of the beaker via the film, drops off the bottom of the beaker and returns to the bath (*after* Daunt and Mendelssohn 1939).

8.1. *The Thickness of the Film*

The thickness of the film below the lambda point was first measured by Daunt and Mendelssohn (1939 b) and Kikoin and Lazarew (1938), who determined the volume of liquid in the film covering a known large area. These authors obtained a value of about 3×10^{-6} cm, essentially independent of temperature, and this result has since been confirmed several times. All the earlier experiments also led to the conclusion that above the lambda point the thickness of the film became quite small, about 10^{-7} cm, a result apparently inconsistent with the suggestion of Frenkel (1940) and Schiff (1941) that the film owes its existence to van der Waals forces, which will presumably show little change in passing through the lambda point. However, both Frenkel and Schiff pointed out that although a film of helium II would be able to establish itself very quickly because of its high mobility, the same would not be true of helium I. In fact, it seems that great precautions have to be observed in order to obtain the true equilibrium thickness of the film ; any extraneous influx of heat may evaporate the film at a rate greater than it can be replaced. Moreover, as Meyer (1954) pointed out, a very small reduction in pressure to just below the saturation value will lead to a very large reduction in the thickness of the film, and the excess helium will drop off very rapidly. However, on restoring the pressure to its original value, the film above the lambda point is only able to build up to its original thickness very slowly ; it is not superfluid and its low thermal conductivity will prevent it discharging the high heat of condensation which must accompany the formation of liquid from the vapour phase. Thus it is not perhaps surprising that there are considerable discrepancies between different observations of the film. As mentioned above, nearly all the earlier work suggested that the film became quite thin above the lambda point; but the recent work of Ham and Jackson (1954) has shown that, by taking sufficient precautions against heat influxes and temperature fluctuations, a thick film is also observed above the lambda point. Thus although the position is still confused, it seems likely that under carefully controlled equilibrium conditions the film is of at least comparable thickness above and below the lambda point, and this is confirmed by measurements of the absorption isotherms (see, for example, Long and Meyer 1953).

Finally, according to Schiff and Frenkel, if the van der Waals forces obey the usual inverse sixth power law, then the thickness of the film should vary as $h^{-1/3}$, where h is the height above the surface of the liquid. Most workers have obtained results closer to an $h^{-1/2}$ relationship but with considerable variations of the absolute magnitude, and it may well be that these variations are due to extraneous heat influxes and other miscellaneous causes. In the former case the incoming heat would probably cause greater evaporation at the top of the film, and the thickness would fall off more quickly than according to the cube root relation (Meyer 1955). Schiff and Frenkel also pointed out that the film thickness should depend on the surface energy of the substrate, but in fact the expected difference even in favourable cases turns out to be within the variations in experimental observations (Smith and Boorse 1955). It must be added, however, that there is still no satisfactory explanation of the report by Ham and Jackson (1954) that if the substrate is contaminated with solid air the film may be considerably thicker, a result in accord with the earlier observations of Bowers and Mendelssohn (1950) that the flow rate in such cases is greatly augmented.

8.2. *Transport Properties*

Below the lambda point the film appears to be sufficiently thick for the properties of the helium in the film to be essentially those of the bulk liquid, and the film exhibits superfluidity and permits of a high heat transport. This superfluidity was studied in detail by Daunt and Mendelssohn (1939 b), who found that the rate of flow was proportional to the perimeter of the vessel; their results are shown in figure 24. The rate of mass transport through the film will be proportional to the velocity of flow, to the thickness of the film and to the fraction of the helium free to move, that is to the superfluid density. As discussed in § 7.2, one would expect the velocity of flow to be equal to the critical velocity which will in turn depend on the thickness of the film. Figure 24 shows that the rate of flow varies with temperature in much the same way as the density of the superfluid does; this implies that the product of film thickness and critical velocity is independent of temperature. Therefore, as the film thickness is approximately independent of temperature at least below the lambda point, the critical velocity appears to be a function of film thickness which is at least approximately independent of temperature, a result which is not in disagreement with the conclusions of § 7.3. (It should be mentioned that as the thickness of the film varies with height there is the very considerable difficulty that it is not really clear which value of the thickness is to be taken as characteristic of the critical velocity.) We also note that both Ambler and Kurti (1952) and Waring (1955) have reported a considerable increase in the rate of film transport below about 0·6°K. This result does not fit into the general picture, and it would be interesting to confirm it by experiments under rather more isothermal conditions than have hitherto been possible.

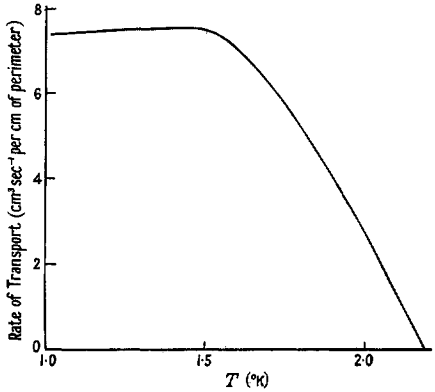

Figure 24. The rate of transport of helium II through the film, as a function of temperature (*after* Daunt and Mendelssohn).

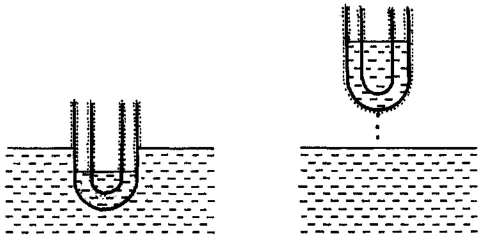

Figure 25. The double-beaker experiment. The helium in the beakers flows through the film to the lowest gravitational level ; note that the flow takes place between the inner and outer beakers even though any difference in the levels is imperceptible (*after* Daunt and Mendelssohn 1946).

One of the most significant of the film flow experiments is that of Daunt and Mendelssohn (1946) which is illustrated in figure 25 ; on lifting the double beaker assembly from the helium bath, the liquid from both inner and outer beakers ran out into the bath. In particular flow took place from the inner to outer beaker, although no pressure or temperature gradient could be observed between them. Likewise, on lowering the beaker into the liquid, a flow occurred similarly but in the opposite direction. This behaviour implies that the kinetic energy of the moving liquid in the film is not dissipated on entering the bulk liquid but must be converted into free energy, that is to say the flow of the liquid must be purely potential flow.

8.3. *The Unsaturated Film*

A comparatively large number of experiments have been made on the unsaturated film, that is to say film in equilibrium with unsaturated helium vapour and therefore not so thick as the saturated film. The results of these experiments are not easy to interpret, for apparently quite similar experiments do not always give the same results. At present it hardly seems possible to correlate all the observations into a coherent system, and it seems that some unaccounted factors may be influencing at least some of the experiments. The position has been reviewed by Long and Meyer (1953) ; here we only mention that the lambda anomaly in the specific heat is smeared out in thin films and occurs at a lower temperature (Frederikse 1949) ; the temperature at which superfluidity occurs is likewise depressed (Bowers, Brewer and Mendelssohn 1951, Long and Meyer 1955). This shift in the lambda temperature as the thickness of the film approaches atomic dimensions has been briefly discussed by Ziman (1953 b), and is probably akin to the depression of the onset of superfluidity observed for the flow of helium through narrow channels of the order of 5×10^{-7} cm diameter (Atkins, Seki and Condon 1956).

§ 9. THE PERMANENT LIQUIDS

We have seen that an essential aspect of the behaviour of helium II is that it remains liquid down to very low temperatures where quantum effects are important. However, helium I (the modification of helium above the lambda point), the isotope ^{3}He, and mixtures of ^{3}He and ^{4}He also exist as liquids at these temperatures and have very different properties from helium II. Thus any satisfactory theory of helium II must indicate whether or not these other ' permanent ' liquids behave like helium II. Therefore we now give a short account of their properties, and then discuss whether their differences from helium II can be predicted from the present theory.

9.1. *Liquid Helium I*

Liquid ^{4}He above 2·17°K (helium I) does not exhibit superfluidity, a high transport of heat or any other of the phenomena associated with the two-fluid model, and is not very different from most other liquids. In contrast to helium II the specific heat has a value which only varies slightly with temperature (Keesom and Clusius 1932), and the expansion coefficient is positive. Viscous processes and heat flow may be described in the usual way by coefficients of viscosity (de Troyer, van Itterbeek and van den Berg 1951) and thermal conductivity (Bowers 1952), and the absorption of sound is entirely accounted for by the classical expression involving these two coefficients. As in the case of helium II, the large influence of the zero-point energy results in the molar volume being very much greater than one would expect from classical theory. The atoms are comparatively free in this large volume and the behaviour of helium I is in some ways more like that of a gas than a liquid ; for instance, the coefficient of viscosity is small and *decreases* with falling temperature. However, the experiments of Tjerkstra (1952) show that if the helium is compressed to higher densities, thereby reducing the space available for each atom, the viscosity eventually rises with falling temperature as in other liquids (figure 26). The thermal conductivity also has a small value which decreases with falling temperature, and is related reasonably well to the coefficient of viscosity by the usual gas kinetic expression $k = 2{\cdot}5\eta$ (Grenier 1951). Virtually all the differences between helium I and helium II may be accounted for by the fact that helium I does not exhibit any of the phenomena characteristic of the two-fluid model, and it is satisfactory to see that this result is inherent in the theory of helium II. It is an essential feature of the two-fluid model that the normal density ρ_n is less than the total density ρ, and also that ρ_n increases steadily with temperature, thus at some temperature ρ_n must become equal to ρ; there is then no superfluid component and the whole concept of the two-fluid model becomes invalid. Indeed, the lambda point may be characterized by $(\rho_n)_\lambda = \rho$.

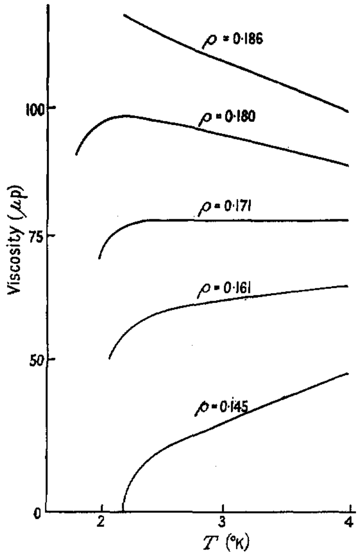

Figure 26. The coefficient of viscosity of liquid helium I as a function of density. At low densities the viscosity decreases with falling temperature as in a gas, but at higher densities a more liquid-like behaviour is observed (*after* Tjerksta 1953).

9.2. *Liquid* ^{3}He

Liquid ^{3}He was first observed by Sydoriak, Grilly and Hammel (1949), and found to have a critical point at 3·34°K in agreement with the value previously predicted by de Boer and Lunbeck (1948) using a reduced quantum-mechanical equation of state. The density of the liquid is very low (about 0·08 g cm^{-3} at 1°K compared with 0·14 g cm^{-3} for ^{4}He), indicating the large role played by the zero-point energy. As in the case of ^{4}He, the liquid does not solidify under its vapour pressure, at least down to 0·5°K (Osborne, Abraham and Weinstock 1952), and a pressure of at least 30 atmospheres must be applied to obtain the solid phase. The adsorption isotherms (Hoffman, Edeskuty and Hammel 1956) are very like those for ^{4}He. Thus it appears that the interatomic forces in liquid ^{3}He and ^{4}He are quite similar and give rise in both cases to liquids with a very open structure which remain fluid down to the lowest temperatures.

At present the number of experiments reported on ^{3}He is small compared with the very large number on ^{4}He, but it is already clear that the lighter isotope has quite different properties. Measurements of the specific heat and entropy give no indication of any lambda or other type of transition down to 0·3°K (Osborne, Abraham and Weinstock 1954, Roberts and Sydoriak 1954, de Vries and Daunt 1954). Weinstock, Osborne and Abraham (1949) using a flow-type experiment have shown that the viscosity has a quite normal value of about 25 μp and that there is no evidence of superfluidity down to 1·05°K. Daunt and Heer (1950) showed that the addition of ^{3}He to ^{4}He depresses the lambda point as defined by the onset of superfluid flow in the film ; a solution containing 89% of ^{3}He has a lambda temperature of 0·38° and it appears likely that pure ^{3}He will not give rise to a superfluid film at all. Thus it seems well established that liquid ^{3}He does not exhibit the phenomena characteristic of the two-fluid model.

For lack of a better model the properties of ^{3}He have been discussed on the assumption that it behaves rather as an ideal Fermi–Dirac gas, even though the limitations on regarding ^{4}He as an ideal Bose gas are known to be considerable (§ 9.4). The vapour pressure and the specific heat measurements are in quite fair agreement with such a model and one would thus predict a degeneracy temperature of about 5°K. However, this result is at variance with measurements of the magnetic susceptibility arising from the nuclear spin of the ^{3}He atoms ; this susceptibility obeys Curie's law down to about 0·5° K (W. M. Fairbank, Ard and Walters 1954) indicating that it is only at about this temperature that degeneracy sets in. The present position, which has been reviewed by Hammel (1955), is obscure and will not be considered further, it being sufficient here to note that the properties of ^{3}He are substantially different from those of ^{4}He.

9.3. *Liquid* ^{3}He–^{4}He *Mixtures*

At first sight it might appear superfluous to treat ^{3}He–^{4}He mixtures as distinct forms of permanent liquids, but in fact some of their properties differ markedly from those of either pure isotope. Owing to the relative scarcity of ^{3}He most experiments have been made with solutions containing only a small percentage of this isotope, but as we are concerned principally with the properties of ^{4}He this is of no great consequence. At low concentrations, the ^{3}He atoms will be virtually independent of each other and will not exhibit any of the bulk properties of ^{3}He, they rather act as a solute or impurity (Landau and Pomeranchuk 1948). Thus they give rise to an osmotic pressure whose magnitude is determined by the law of van't Hoff, and which may be developed between two vessels joined by a very fine slit which is permeable only to superfluid ^{4}He (Taconis, Beenakker and Dokoupil 1950). According to the nature of the two-fluid components of helium II, it is to be expected that solute atoms will interact only with the excitations, and will therefore be carried along with the normal fluid (Landau and Pomeranchuk 1948, Khalatnikov 1952 d). Thus in the presence of a temperature gradient all the ^{3}He is carried to the cold part of the apparatus by the flow of normal fluid, and this is the basis of the heat flush method of concentrating ^{3}He from ^{3}He–^{4}He mixtures (Lane, H. A. Fairbank, Aldrich and Nier 1948). The association of the ^{3}He with the normal fluid is confirmed by Andronikashvili disc measurements in ^{3}He–^{4}He mixtures (Pellam 1955) ; the density of the normal fluid does not tend towards zero at low temperatures but appears to take up a constant value equal to about twice the mass of the ^{3}He atoms present in unit volume (figure 27). This increase in ρ_n indicates that the ^{3}He atoms are dragged round with the discs ; that the magnitude of the increase is about twice the mass density of the ^{3}He atom may be accounted for by the fact that the moving atoms have to push aside ^{4}He atoms and hence have a greater inertial resistance than their own mass (Feynman 1954). Pellam also obtained semi-qualitative values for the viscosity of the normal component, and found that the value of η_n no longer increased below 1·4°K as in pure ^{4}He. The viscosity of the pure liquid rises because the decreasing number of rotons results in a longer mean free path for the phonons and a correspondingly greater transport of momentum. The additional ^{3}He atoms act as a comparatively large number of scattering centres, the mean free path remains constant, and the viscosity decreases steadily with falling temperature as the number of phonons decreases.

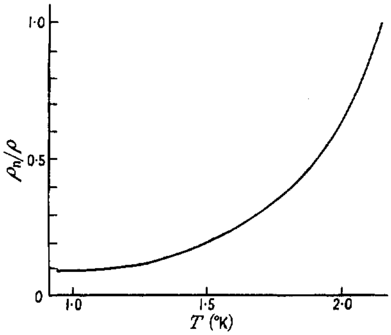

Figure 27. The normal density ρ_n of a 3·3% ^{3}He–^{4}He solution, as measured by Andronikashvili's method. At low temperatures the density does not tend towards zero but takes up a value of the order of the effective density of the ^{3}He (*after* Pellam 1955).

At first sight it might appear that a small percentage of ^{3}He atoms in liquid ^{4}He would form scattering centres which would considerably reduce the transport of heat in the liquid. However, these impurity atoms are not fixed in space as they would be in a solid, but are free to move with the normal fluid. Thus as the high heat transport in liquid ^{4}He is due to the relative motion of the two fluids, it will not be much affected if some ^{3}He atoms are carried along with the normal fluid. This is well shown by the experiments of Lynton and H. A. Fairbank (1950), who observed that a pulse of heat is not appreciably attenuated by the addition of small amounts of ^{3}He. It is to be noted that the above discussion assumes that the ^{3}He atoms are distributed uniformly through the liquid, but in fact they will tend to diffuse to the colder parts of the apparatus and form a concentration gradient there. In the experiments of Lynton and Fairbank, a uniform concentration of ^{3}He was ensured by the use of single pulses in order to keep the total heat flow very low, but if an attempt is made to measure heat flows *under steady uni-directional conditions*, then a concentration gradient of ^{3}He will be set up. In measurements of this type, the concentration gradient near the colder surface will prevent the free movement of ^{3}He atoms, and forces will be set up tending to restrain the motion of the normal fluid. Thus the measured heat flow is reduced from the very high value typical of ^{4}He to a quite moderate value (Beenakker, Taconis, Lynton, Dokoupil and van Soest 1952). Finally, as shown in figure 28, the addition of a small amount of ^{3}He (say 1%) to liquid ^{4}He will permit the liquid to transmit sharp pulses of temperature waves down to at least 0·2°K (King and H. A. Fairbank 1954). As discussed in § 6.3, temperature waves are not observed in pure ^{4}He below about 0·7°K because there are so few interactions between the excitations that equilibrium in the normal fluid cannot be established in distances of the order of a wavelength. However, even for a 1% concentration, the ^{3}He atoms are quite close together and, as is shown by the heat flush experiments, they interact with the excitations ; thus equilibrium may be established in the normal fluid and temperature waves observed. King and Fairbank also showed that the presence of ^{3}He atoms has a large effect on the velocity of the waves (figure 29), because the ^{3}He changes the values of both ρ_n and the specific heat and entropy of the liquid (Pomeranchuk 1949). At very low temperatures the normal fluid consists almost entirely of ^{3}He atoms and a temperature wave is essentially a variation in the density of these atoms which drag the phonons along with them. In this region the velocity of the waves should approximate to that of a sound wave in a gas of ^{3}He atoms, and therefore be of the order of the velocity of the ^{3}He atoms, that is $(3kT/\mu)^{1/2}$ where μ is the effective mass (Feynman 1954). A more exact expression has been given by Pomeranchuk, namely $(5kT/3\mu)^{1/2}$, and although this does not appear to be in quantitative agreement with the measured values it is at least qualitatively correct.

Figure 28. The shape of heat pulses through liquid helium and viewed on an oscilloscope trace : (*a*) pure ^{4}He at 0·45°K, (*b*) ^{4}He containing 0·32% ^{3}He at 0·45°K (*after* King and H. A. Fairbank 1954).

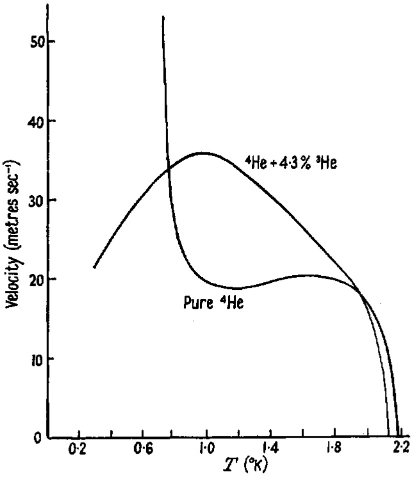

Figure 29. The velocity of temperature waves in ^{3}He–^{4}He solutions as a function of temperature (*after* King and H. A. Fairbank 1954).

As mentioned previously, the lambda point is depressed by the addition of ^{3}He ; this may be accounted for by the fact that the ^{3}He atoms increase the effective density of normal fluid, and therefore the criterion for the lambda point $[(\rho_n)_\lambda = \rho]$ is attained at lower temperatures. Measurements have also been made on the specific heat of mixtures, on the heat of mixing, and on the relative concentration of the two isotopes in the gaseous and liquid phases. These last experiments have been reviewed in detail (Daunt 1952, Beenakker and Taconis 1955), but do not at present seem to throw much light on the properties of pure ^{4}He.

9.4. *Statistics and the Lambda Point*

We have seen that dilute solutions of ^{3}He in helium II behave in a manner which may be predicted by an extension of the theory of helium II; however, the same is not true of ^{3}He itself, and it seems that this is due to the influence of statistics. Thus ^{3}He presumably obeys Fermi statistics, whereas the theory of ^{4}He assumes that the particles obey Bose statistics, and that permutations of the atoms leave the wave function quite unchanged. It is significant that if one calculates the quantity ρ_n of equation (6) (the normal density of the two-fluid model) for Bose–Einstein and Fermi–Dirac ideal gases, one finds that ρ_n is equal to the total density except for a Bose–Einstein gas below the ' condensation ' temperature. Thus the essential feature of the two-fluid model appears to be a characteristic of an assembly obeying Bose statistics. We may also mention here that de Boer and Cohen (1955) have discussed a cell model in which each cell contains more than one atom, and in which exchange effects are therefore important. It is by no means clear how this model is to be associated with the present theory; but for a cell model with four atoms in each cell the statistics are responsible for a very large difference in the entropy of the two isotopes, such as is observed (figure 30).

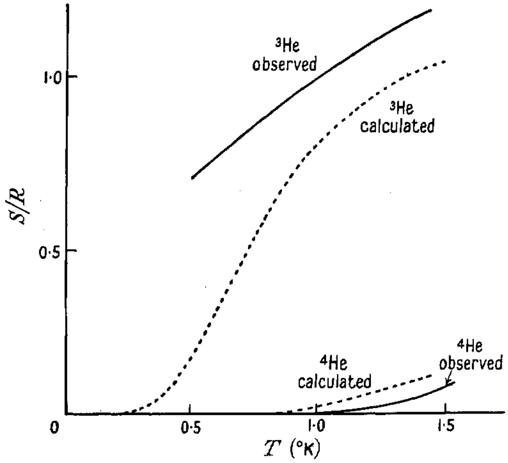

Figure 30. The entropy of ^{3}He and ^{4}He as calculated using a cell model with four atoms in each cell (*after* de Boer and Cohen 1955).

The importance of statistics was first stressed by F. London (1938), who pointed out that an ideal gas of the same density as liquid helium, and obeying Bose statistics, should exhibit an unusual type of phase transition at 3·14°K, with a specific heat anomaly somewhat similar to that observed in the liquid. He therefore suggested that the lambda transition was the analogy in the liquid of the ' condensation ' in the perfect gas, and on this basis Tisza was able to predict the existence of temperature waves. However, any such analogy cannot be a very close one, for liquid helium is not an ideal gas but an assembly of real atoms packed together as closely as they can get. From the specific heat and thermal conductivity measurements below 0·6°K, it is clear that there is a contribution to the thermal energy akin to that in a Debye type solid, and this certainly cannot arise in an ideal gas. The temperature of the lambda point *decreases* with increasing density of the liquid, in contrast to what would be expected for an ideal gas. The observed scattering cross section of helium II for neutrons (Sommers, Dash and Goldstein 1955) is quite different from that calculated using the gas model (Goldstein, Sweeney and Goldstein 1950), and the scattering of light is much less (Jakovlev 1943) than is predicted (Galanin 1940, Schiff 1941). Finally, as discussed in §§ 5.3 and 5.5, Tisza's theory leads to quite incorrect values for the fountain pressure and the velocity of temperature waves at low temperatures.

Although the ideal gas treatment is plainly inadequate, Feynman (1953 a) has shown that a form of Bose condensation is to be expected even in an assembly of strongly interacting particles. Feynman proceeds by writing down the complete partition function for the liquid, which he then simplifies and evaluates by approximate methods. Even the simplified function is difficult to evaluate, and Feynman concluded that the predicted transition was third order rather than second ; however, Chester (1955) claims that a more accurate evaluation of the partition function leads to a second-order transition as is observed. It is typical of the difficulties in carrying out this evaluation that it is not entirely clear whether Feynman's partition function also leads to a liquid–gas transition of the normal type ; see, for example, ter Haar (1954) and Chester (1955), who advance opposing views. We note also that Feynman predicts that the specific heat begins to rise in the helium I region as the lambda point is approached, as is observed.

§ 10. CONCLUSION

To sum up the present position we can hardly do better than quote Landau (1949) : " It follows unambiguously from quantum mechanics that for every slightly excited macroscopic system a conception can be introduced of ' elementary excitations ', which describe the ' collective ' motion of the particles and which have certain energies ϵ and momenta p (leaving aside the question as to the actual dependence $\epsilon(p)$, i.e. the actual form of the energy spectrum). It is this assumption, indisputable in my opinion, which is the basis of the microscopical part of my theory. On the contrary, every consideration of the motion of individual atoms in the system of strongly interacting particles is in contradiction with the first principles of quantum mechanics." The consideration of the role of the *excitations* rather than of the *atoms* is the distinctive contribution made by Landau to the theory of helium II ; the ' normal fluid ' is an aggregate of excitations rather than of excited atoms, as postulated by Tisza. It is for this reason that the phonons which are responsible for the behaviour of second sound and the fountain effect at low temperatures enter naturally into Landau's theory but not into Tisza's.

REFERENCES

ALLCOCK, G. R., and KUPER, C. G., 1955, *Proc. Roy. Soc.* A, **231**, 226.

ALLEN, J. F., and JONES, H., 1938, *Nature, Lond.*, **141**, 243.

ALLEN, J. F., and MISENER, A. D., 1939, *Proc. Roy. Soc.* A, **172**, 467.

AMBLER, E., and KURTI, N., 1952, *Phil. Mag.*, **43**, 260.

ANDRONIKASHVILI, E. L., 1948, *J. Exp. Theor. Phys.*, **18**, 424, 429 ; 1949, *Ibid.*, **19**, 535.

ANDRONIKASHVILI, E. L., and KAVERKIN, I. P., 1955, *J. Exp. Theor. Phys.*, **28**, 126.

ATKINS, K. R., 1952, *Advanc. Phys.* (*Phil. Mag.* Suppl.), **1**, 169 ; 1953, *Phys. Rev.*, **89**, 526 ; 1955, *Ibid.*, **98**, 319.

ATKINS, K. R., and CHASE, C. E., 1951, *Proc. Phys. Soc.* A, **64**, 826.

ATKINS, K. R., and EDWARDS, M. H., 1955, *Phys. Rev.*, **97**, 1429.

Atkins, K. R., and Hart, K. H., 1954, *Canad. J. Phys.*, **32**, 381.

Atkins, K. R., and Osborne, D. V., 1950, *Phil. Mag.*, **41**, 1078.

Atkins, K. R., Seki, H., and Condon, E. U., 1956, *Phys. Rev.*, **102**, 582.

Atkins, K. R., and Stasior, R. A., 1953, *Canad. J. Phys.*, **31**, 1156.

Beenakker, J. J. M., and Taconis, K. W., 1955, *Progress in Low Temperature Physics* (Ed. C. J. Gorter) (Amsterdam : North-Holland), p. 108.

Beenakker, J. J. M., Taconis, K. W., Lynton, E. A., Dokoupil, Z., and van Soest, G., 1952, *Physica*, **18**, 433.

de Boer, J., and Cohen, E. G. D., 1955, *Physica*, **21**, 79.

de Boer, J., and Lunbeck, R. J., 1948, *Physica*, **14**, 510.

Bots, G. J. C., and Gorter, C. J., 1956, *Physica*, **22**, 503.

Bowers, R., 1952, *Proc. Phys. Soc.* A, **65**, 511.

Bowers, R., Brewer, D. F., and Mendelssohn, K. M., 1951, *Phil. Mag.*, **42**, 1445.

Bowers, R., and Mendelssohn, K. M., 1950, *Proc. Phys. Soc.* A, **63**, 1318.

Brewer, D. F., Edwards, D. O., and Mendelssohn, K. M., 1955, *Proc. Phys. Soc.* A, **68**, 939.

Casimir, H. B. G., 1938, *Physica*, **5**, 495.

Chandrasekhar, B. S., and Mendelssohn, K. M., 1955, *Proc. Phys. Soc.* A, **68**, 857.

Chase, C. E., 1953, *Proc. Roy. Soc.* A, **220**, 116.

Chase, C. E., and Herlin, M. A., 1955, *Phys. Rev.*, **97**, 1447.

Chester, G. V., 1955, *Phys. Rev.*, **100**, 455.

Dash, J. G., 1954, *Phys. Rev.*, **94**, 825.

Daunt, J. G., 1952, *Advanc. Phys.* (*Phil. Mag.* Suppl.), **1**, 209.

Daunt, J. G., and Heer, C. V., 1950, *Phys. Rev.*, **79**, 46.

Daunt, J. G., and Mendelssohn, K. M., 1939 a, *Nature, Lond.*, **143**, 719 ; 1939 b, *Proc. Roy. Soc.* A, **170**, 423, 439 ; 1946, *Nature, Lond.*, **157**, 839.

Daunt, J. G., and Smith, R. S., 1954, *Rev. Mod. Phys.*, **26**, 172.

Dingle, R. B., 1949, *Proc. Phys. Soc.* A, **62**, 648 ; 1952 a, *Advanc. Phys.* (*Phil. Mag.* Suppl.), **1**, 111 ; 1952 b, *Proc. Phys. Soc.* A, **65**, 374.

Fairbank, H. A., and Wilks, J., 1955, *Proc. Roy. Soc.* A, **231**, 545.

Fairbank, W. M., Ard, W. B., and Walters, G. K., 1954, *Phys. Rev.*, **95**, 566.

Feynman, R. P., 1953 a, *Phys. Rev.*, **91**, 1291 ; 1953 b, *Ibid.*, **91**, 1301 ; 1954, *Ibid.*, **94**, 262 ; 1955, *Progress in Low Temperature Physics* (Ed. C. J. Gorter) (Amsterdam : North-Holland), p. 17.

Feynman, R. P., and Cohen, M., 1956, *Phys. Rev.*, **102**, 1189.

Franchetti, S., 1954, *Nuovo Cim.*, **12**, 743.

Frederikse, H. P. R., 1949, *Physica*, **15**, 860.

Frenkel, J., 1936, *Wave Mechanics, Elementary Theory* (Oxford : University Press) ; 1940, *J. Phys., Moscow*, **2**, 365.

Galanin, A., 1940, *J. Exp. Theor. Phys.*, **10**, 1267.

Ginsburg, V. L., 1955, *J. Exp. Theor. Phys.*, **29**, 244, *Soviet Physics*, **2**, 170.

Goldstein, L., Sweeney, D., and Goldstein, M., 1950, *Phys. Rev.*, **77**, 319 ; **80**, 141.

Gorter, C. J., 1952, *Phys. Rev.*, **88**, 681.

Gorter, C. J., and Mellink, J. H., 1949, *Physica*, **15**, 285.

Grenier, C., 1951, *Phys. Rev.*, **83**, 598.

ter Haar, D., 1954, *Phys. Rev.*, **95**, 895.

Hall, H. E., 1954, *Proc. Phys. Soc.* A, **67**, 485 ; 1955, *Conférence de Physique des Basses Températures, Paris* (Paris : Institut International du Froid), p. 63.

Hall, H. E., and Vinen, W. F., 1955, *Conférence de Physique des Basses Températures, Paris* (Paris : Institut International du Froid), p. 66.

Ham, A. C., and Jackson, L. C., 1954, *Phil. Mag.*, **45**, 1084.

Hammel, F., 1955, *Progress in Low Temperature Physics* (Ed. C. J. Gorter) (Amsterdam : North-Holland), p. 78.

Heikkila, W. J., and Hollis-Hallett, A. C., 1955, *Canad. J. Phys.*, **33**, 420.

Henshaw, D. G., and Hurst, D. G., 1953, *Phys. Rev.*, **91**, 1222.

Hercus, G. R., and Wilks, J., 1955, *Phil. Mag.*, **45**, 1163.

Hoffman, C. J., Edeskuty, F. J., and Hammel, E. F., 1956, *J. Chem. Phys.*, **24**, 124.

Hollis-Hallett, A. C., 1950, *Proc. Phys. Soc.* A, **63**, 1367 ; 1952, *Proc. Roy. Soc.* A, **210**, 404 ; 1955, *Progress in Low Temperature Physics* (Ed. C. J. Gorter) (Amsterdam : North-Holland), p. 64.

Hung, C. S., Hunt, B., and Winkel, P., 1952, *Physica*, **18**, 629.

Jakovlev, I. A., 1943, *J. Phys., Moscow*, **7**, 307.

Kapitza, P. L., 1941 a, *J. Exp. Theor. Phys.*, **11**, 581 ; *J. Phys., Moscow*, **5**, 59 ; 1941 b, *J. Exp. Theor. Phys.*, **11**, 1 ; *J. Phys., Moscow*, **4**, 181.

Karim, S. M., and Rosenhead, L., 1952, *Rev. Mod. Phys.*, **24**, 108.

Keesom, W. H., and Clusius, K., 1932, *Proc. Acad. Sci. Amst.*, **35**, 307, *Common. Leider.* No. 219e.

Keesom, W. H., and MacWood, G. E., 1938, *Physica*, **5**, 737.

Keesom, W. H., and Taconis, K. W., 1938 a, *Physica*, **5**, 270 ; 1938 b, *Ibid.*, **5**, 161.

Khalatnikov, I. M., 1950, *J. Exp. Theor. Phys.*, **20**, 243 ; 1952 a, *Ibid.*, **23**, 8 ; 1952 b, *Ibid.*, **23**, 21 ; 1952 c, *Ibid.*, **23**, 253 ; 1952 d, *Ibid.*, **23**, 169.

Kikoin, A. K., and Lazarew, B. G., 1938, *Nature, Lond.*, **142**, 289.

King, J. C., and Fairbank, H. A., 1954, *Phys. Rev.*, **93**, 21.

de Klerk, D., Hudson, R. P., and Pellam, J. R., 1954, *Phys. Rev.*, **93**, 28.

Kolm, H. H., and Herlin, M. A., 1956, *Phys. Rev.*, **102**, 607.

Kramers, H. A., 1952, *Physica*, **18**, 653.

Kramers, H. C., van Peski-Tinbergen, T., Wiebes, J., van den Burg, F. A. W., and Gorter, C. J., 1954, *Physica*, **20**, 743.

Kramers, H. C., Wasscher, J. D., and Gorter, C. J., 1952, *Physica*, **18**, 329.

Kronig, R., 1953, *Physica*, **19**, 535.

Kronig, R., and Thellung, A., 1952, *Physica*, **18**, 749.

Landau, L. D., 1941, *J. Exp. Theor. Phys.*, **11**, 592 ; *J. Phys., Moscow*, **5**, 71 ; 1944, *J. Exp. Theor. Phys.*, **14**, 112 ; *J. Phys., Moscow*, **8**, 1 ; 1947, *Ibid.*, **11**, 91 ; 1949, *Phys. Rev.*, **75**, 884.

Landau, L. D., and Khalatnikov, I. M., 1949, *J. Exp. Theor. Phys.*, **19**, 637, 709.

Landau, L. D., and Lifshitz, E., 1955, *Dokl. Akad. Nauk., S.S.S.R.*, **100**, 669.

Landau, L. D., and Pomeranchuk, I., 1948, *Dokl. Akad. Nauk., S.S.S.R.*, **59**, 669.

Lane, C. T., Fairbank, H. A., Aldrich, L. T., and Nier, A. O., 1948, *Phys. Rev.*, **73**, 256.

Lifshitz, E., 1944, *J. Exp. Theor. Phys.*, **14**, 116 ; *J. Phys., Moscow*, **8**, 110.

Lokken, J. E., 1954, *Canad. J. Phys.*, **32**, 359.

London, F., 1938, *Nature, Lond.*, **141**, 643 ; 1945, *Rev. Mod. Phys.*, **17**, 310 ; 1954, *Superfluids*, Vol. 2 (New York : Wiley).

London, F., and Zilsel, P. R., 1948, *Phys. Rev.*, **74**, 1148.

London, H., 1939, *Proc. Roy. Soc.* A, **171**, 484 ; 1946, *Proceedings of the Cambridge Low Temperature Conference* (London : Physical Society), p. 48.

Long, E., and Meyer, L., 1953, *Advanc. Phys.* (*Phil. Mag.* Suppl.), **2**, 1 ; 1955, *Phys. Rev.*, **98**, 1616.

Lynton, E. A., and Fairbank, H. A., 1950, *Phys. Rev.*, **80**, 1043.

Maurer, R. D., and Herlin, M. A., 1951, *Phys. Rev.*, **81**, 444.

Meyer, L., 1954, *Phys. Rev.*, **93**, 655 ; 1955, *Ibid.*, **97**, 22.

Meyer, L., and Mellink, J. H., 1947, *Physica*, **13**, 197.

Newell, J. A., and Wilks, J., 1956, *Phil. Mag.*, **1**, 588.

Onsager, L., 1949, *Nuovo Cim.*, **6**, Suppl. **2**, 249.

Osborne, D. V., 1951, *Proc. Phys. Soc.* A, **64**, 114.

Osborne, D. W., Abraham, B. M., and Weinstock, B., 1951, *Phys. Rev.*, **82**, 263 ; 1952, *Ibid.*, **85**, 158 ; 1954, *Phys. Rev.*, **94**, 202.

Pellam, J. R., 1949, *Phys. Rev.*, **75**, 1183 ; 1955, *Conférence de Physique des Basses Températures, Paris* (Paris : Institut International du Froid), p. 24.

Pellam, J. R., and Hanson, W. B., 1952, *Phys. Rev.*, **85**, 216 ; 1954, *Ibid.*, **95**, 321.

Pellam, J. R., and Morse, P. M., 1950, *Phys. Rev.*, **78**, 474.

Pellam, J. R., and Squire, C. F., 1947, *Phys. Rev.*, **72**, 1245.

Peshkov, V., 1946, *J. Exp. Theor. Phys.*, **16**, 1000 ; *J. Phys., Moscow*, **10**, 389 ; *Proceedings of the Cambridge Low Temperature Conference* (London : Physical Society), p. 19 ; 1952, *J. Exp. Theor. Phys.*, **23**, 687 ; 1954, *Ibid.*, **27**, 351.

Pippard, A. B., 1951, *Phil. Mag.*, **42**, 1209.
Pomeranchuk, I., 1949, *J. Exp. Theor. Phys.*, **19**, 42.
Prigogine, I., 1954, *Advanc. Phys.* (*Phil. Mag.* Suppl.), **3**, 131.
Reekie, J., and Hutchison, T. S., 1953, *Phys. Rev.*, **92**, 827.
Roberts, T. R., and Sydoriak, S. G., 1954, *Phys. Rev.*, **93**, 1418.
Rollin, B. V., 1936, *Thesis*, University of Oxford.
Rollin, B. V., and Simon, F. E., 1939, *Physica*, **6**, 219.
Schiff, L., 1941, *Phys. Rev.*, **57**, 839.
Simon, F. E., 1934, *Nature, Lond.*, **133**, 460, 524 ; 1950, *Physica*, **16**, 753.
Simon, F. E., and Swenson, C. A., 1950, *Nature, Lond.*, **165**, 829.
Smith, B., and Boorse, H. A., 1955, *Phys. Rev.*, **98**, 328.
Sommers, H. S., Dash, J. G., and Goldstein, L., 1955, *Phys. Rev.*, **97**, 855.
Sydoriak, S. G., Grilly, E. R., and Hammel, E. F., 1949, *Phys. Rev.*, **75**, 303.
Taconis, K. W., Beenakker, J. J. M., and Dokoupil, Z., 1950, *Phys. Rev.*, **78**, 171.
Tisza, L., 1938, *C.R. Acad. Sci., Paris*, **207**, 1035, 1186.
Tjerkstra, H. H., 1952, *Physica*, **18**, 853.
Thellung, A., 1953, *Physica*, **19**, 217.
de Troyer, A., van Itterbeek, A., and van den Berg, G. J., 1951, *Physica*, **17**, 50.
Vinen, W. F., 1955, *Conférence de Physique des Basses Températures, Paris* (Paris : Institut International du Froid), p. 60.
de Vries, G., and Daunt, J. G., 1953, *Phys. Rev.*, **92**, 1572 ; 1954, *Ibid.*, **93**, 631.
Ward, J. C., and Wilks, J., 1951, *Phil. Mag.*, **42**, 314 ; 1952, *Ibid.*, **43**, 48.
Waring, R. K., 1955, *Phys. Rev.*, **99**, 1704.
Weinstock, B., Osborne, D. W., and Abraham, B. M., 1949, *Proceedings of the International Conference on Physics of Low Temperatures* (M.I.T.), p. 47.
Wheeler, R. G., Blakewood, C. H., and Lane, C. T., 1955, *Phys. Rev.*, **99**, 1667.
Wilks, J., 1953, *Nuovo Cim.*, Suppl., **10**, 509.
Winkel, P., 1955, *Physica*, **21**, 322.
Winkel, P., Broese van Groenou, A., and Gorter, C. J., 1955, *Physica*, **21**, 345.
Winkel, P., Delsing, A. M. G., and Gorter, C. J., 1955, *Physica*, **21**, 312.
Winkel, P., Delsing, A. M. G., and Poll, J. D., 1955, *Physica*, **21**, 331.
Ziman, J. M., 1953 a, *Proc. Roy. Soc.* A, **219**, 257 ; 1953 b, *Phil. Mag.*, **44**, 548 ; 1954, *Ibid.*, **45**, 100.

APPENDIX

Recent work on liquid ^{4}He not mentioned in earlier reviews and not referred to in the present paper

Melting curve and zero point energy
Hurst, D. G., and Henshaw, D. G., 1955, *Canad. J. Phys.*, **33**, 797.
Swenson, C. A., 1953, *Phys. Rev.*, **89**, 538.

Phonons
van den Berg, G. J., van Itterbeek, A., van Aardenne, G. M. V., and Herfkens, J. H. J., 1955, *Physica*, **21**, 860.
Geilikman, B. T., 1953, *Dokl. Akad. Nauk., S.S.S.R.*, **94**, 191.
van Itterbeek, A., and Forrez, G., 1954, *Physica*, **20**, 133.

Rotons
Beaumont, C. F. A., 1954, *Canad. J. Phys.*, **32**, 759.
Brenig, W., 1956, *Z. Phys.*, **144**, 488.
Goldstein, L., 1953, *Phys. Rev.*, **89**, 597.
Houston, W. V., and Rorscach, H. E., 1955, *Phys. Rev.*, **100**, 1003.
Hurst, D. G., and Henshaw, D. G., 1955, *Phys. Rev.*, **100**, 994.
Kaempffer, F. A., 1954, *Canad. J. Phys.*, **32**, 264.

Klemens, P. G., 1955, *Phys. Rev.*, **97**, 1181.
Kuper, C. G., 1955, *Proc. Roy. Soc.* A, **233**, 223.
Matsuda, H., 1954, *Progr. Theor. Phys., Japan*, **12**, 537.
Mikura, Z., 1954, *Progr. Theor. Phys., Japan*, **11**, 207.
Pekeris, C. L., 1953, *Proc. Nat. Acad. Sci., Wash.*, **39**, 443.
Rice, O. K., 1956, *Phys. Rev.*, **102**, 1416.
Temperley, H. N. V., 1953, *Proc. Phys. Soc.* A, **66**, 995 ; 1954, *Ibid.*, **67**, 901.
Tweet, A. G., 1954, *Phys. Rev.*, **93**, 15.
Tyabji, S. F. B., 1953, *Nature, Lond.*, **172**, 849 ; 1954, *Proc. Camb. Phil. Soc.*, **50**, 449.

Superfluid flow
Chandrasekhar, B. S., and Mendelssohn, K., 1953, *Proc. Roy. Soc.* A, **217**, 18.
Lane, C. T., and Dyba, R. V., 1953, *Phys. Rev.*, **92**, 829.

Viscosity
Donnelly, R. J., Chester, G. V., Walmsley, R. H., and Lane, C. T., 1956, *Phys. Rev.*, **102**, 3.
Gamtsemlidze, G. A., 1955, *Dokl. Akad. Nauk., S.S.S.R.*, **100**, 441.
Hollis-Hallett, A. C., 1953, *Proc. Camb. Phil. Soc.*, **49**, 717.

Fountain effect
Forstat, H., and Reynolds, C. A., 1956, *Phys. Rev.*, **101**, 513.
Herival, J. W., 1954, *Nature, Lond.*, **174**, 322.
Manchester, F. D., 1955, *Canad. J. Phys.*, **33**, 146.
van den Meijdenberg, C. J. N., Taconis, K. W., Beenakker, J. J. M., and Wansink, D. H. N., 1954, *Physica*, **20**, 157.
Reynolds, C. A., 1954, *Phys. Rev.*, **93**, 1118.

Heat transport
Inoue, T., 1953, *J. Phys. Soc., Japan*, **8**, 450.
Klemens, P. G., 1955, *Aust. J. Phys.*, **8**, 206.
Kurti, N., and McIntosh, J., 1955, *Phil. Mag.*, **46**, 104.
Mikura, Z., 1954, *Progr. Theor. Phys., Japan*, **11**, 503.

Irreversible processes
Rice, O. K., 1953, *Phys. Rev.*, **89**, 793.

Absorption of pressure and temperature waves
Chase, C. E., 1956, *Amer. J. Phys.*, **24**, 136.
Zinov'eva, K. N., 1953, *J. Exp. Theor. Phys.*, **25**, 235.

Mean free path effects
Osborne, D. V., 1956, *Phil. Mag.*, **1**, 301.

Supercritical flow
Dash, J. G., 1954, *Phys. Rev.*, **94**, 1091 ; 1955, *Ibid.*, **97**, 1038.
Kasuya, T., 1953, *Progr. Theor. Phys., Japan*, **9**, 87, 89, 90.
Swim, R. T., and Rorschach, H. E., 1955, *Phys. Rev.*, **97**, 25.
Wansink, D. H. N., Taconis, K. W., Staas, F. A., and Reuss, J., 1955, *Physica*, **21**, 596.

Rotating helium
Andronikashvili, E. L., 1952, *J. Exp. Theor. Phys.*, **22**, 62.
Blatt, J. M., Butler, S. T., and Schafroth, M. R., 1955, *Phys. Rev.*, **100**, 481.
Butler, S. T., and Blatt, J. M., 1955, *Phys. Rev.*, **100**, 495.
Lifshitz, I. M., and Kaganov, M. I., 1955, *J. Exp. Theor. Phys.*, **29**, 257 ; *Soviet Physics*, **2**, 172.

Film

Franchetti, S., 1955, *Nuovo Cim.*, **2**, 1127.
McCrum, N. G., and Eisenstein, J. C., 1955, *Phys. Rev.*, **99**, 1326.
Rice, O. K., and Widom, B., 1953, *Phys. Rev.*, **90**, 987.

Thickness of the film

Atkins, K. R., 1953, *Phys. Rev.*, **92**, 1571 ; 1954, *Canad. J. Phys.*, **32**, 347.
Bowers, R., 1953, *Phys. Rev.*, **91**, 1016 ; 1953, *Phil. Mag.*, **44**, 1309.
Brewer, D. F., and Mendelssohn, K., 1953, *Phil. Mag.*, **44**, 340.
Jackson, L. C., and Henshaw, D. G., 1953, *Phil. Mag.*, **44**, 14.
Knudsen, W. C., and Dillinger, J. R., 1954, *Phys. Rev.*, **95**, 279.
McCrum, N. G., 1954, *Phil. Mag.*, **45**, 1302.

Transport properties of the film

Beams, W. J., 1955, *Phys. Rev.*, **98**, 1138.
Dyba, R. V., Lane, C. T., and Blakewood, C. H., 1954, *Phys. Rev.*, **95**, 1365.
Esel'son, B. N., and Lazarev, B. G., 1952, *J. Exp. Theor. Phys.*, **23**, 552.
Ham, A. C., and Jackson, L. C., 1953, *Phil. Mag.*, **44**, 1403.
Picus, G. S., 1953, *Phys. Rev.*, **90**, 719 ; 1954, *Ibid.*, **94**, 1459.
Taconis, K. W., 1955, *Phys. Rev.*, **97**, 1176.

The unsaturated film

Bowers, R., 1953, *Phil. Mag.*, **44**, 485.
McCrum, N. G., and Mendelssohn, K., 1954, *Phil. Mag.*, **45**, 102.
Tjerkstra, H. H., Hooftman, F. J., and van den Meydenberg, C. J. N., 1953, *Physica*, **19**, 935.
White, D., Chou, C., and Johnston, H. L., 1952, *J. Chem. Phys.*, **20**, 1819.

^{3}He–^{4}He solutions

Guttman, L., and Arnold, J. R., 1953, *Phys. Rev.*, **92**, 547.
Khalatnikov, I. M., 1952, *J. Exp. Theor. Phys.*, **23**, 264.

The nature of the lambda transition

Blatt, J. M., and Butler, S. T., 1954, *Phys. Rev.*, **96**, 1149 ; 1955, *Ibid.*, **100**, 476.
Buckthought, K., 1953, *Canad. J. Phys.*, **31**, 932.
Chester, G. V., 1954, *Phys. Rev.*, **93**, 1412, **94**, 246.
Franchetti, S., 1955, *Nuovo Cim.*, **12**, 743, *Ibid.*, Suppl. **1**, 159.
Goldstein, L., 1955, *Phys. Rev.*, **100**, 981.
Goldstein, L., and Sommers, H. S., 1956, *Phys. Rev.*, **101**, 1235.
Hurst, C., 1955, *Proc. Phys. Soc.* B, **68**, 521.
Kikuchi, R., 1954, *Phys. Rev.*, **96**, 563.
Lawson, A. W., and Meyer, L., 1954, *Phys. Rev.*, **93**, 259.
van der Leeden, P., 1955, *Physica*, **21**, 561.
Price, P. J., 1954, *Phys. Rev.*, **94**, 257.
Rice, O. K., 1954, *Phys. Rev.*, **93**, 1161, **96**, 1460.
Zilsel, P. R., 1953, *Phys. Rev.*, **92**, 1106.

Miscellaneous

Atkins, K. R., 1953, *Canad. J. Phys.*, **31**, 1165.
Berman, R., and Poulter, J., 1952, *Phil. Mag.*, **43**, 1047.
Clement, J. R., Logan, J. K., and Gaffney, J., 1955, *Phys. Rev.*, **100**, 743.