Lesson 03 - Projectile Motion I

How to make parametric plots

In[ ]:= ? ParametricPlot

Out[ ]=

Symbol

ParametricPlot[{f_x, f_y}, {u, u_{min}, u_{max}}] generates a parametric plot of a curve with x and y coordinates f_x and f_y as a function of u.

ParametricPlot[{{f_x, f_y}, {g_x, g_y}, …}, {u, u_{min}, u_{max}}] plots several parametric curves.

ParametricPlot[{f_x, f_y}, {u, u_{min}, u_{max}}, {v, v_{min}, v_{max}}] plots a parametric region.

ParametricPlot[{{f_x, f_y}, {g_x, g_y}, …}, {u, u_{min}, u_{max}}, {v, v_{min}, v_{max}}] plots several parametric regions.

ParametricPlot[{…, w[{f_x, f_y}], …}, …] plots the curve {f_x, f_y} with features defined by the symbolic wrapper w.

ParametricPlot[…, {u, v} ∈ *reg*] takes parameters {u, v} to be in the geometric region *reg*.

How to solve a differential equation numerically

In[ ]:= ? NDSolveValue

Out[ ]=

Symbol

NDSolveValue[*eqns*, *expr*, {x, x_{min}, x_{max}}] gives the value of *expr* with functions determined by a numerical solution to the ordinary differential equations *eqns* with the independent variable x in the range x_{min} to x_{max}.

NDSolveValue[*eqns*, *expr*, {x, x_{min}, x_{max}}, {y, y_{min}, y_{max}}] solves the partial differential equations *eqns* over a rectangular region.

NDSolveValue[*eqns*, *expr*, {x, y} ∈ Ω] solves the partial differential equations *eqns* over the region Ω.

NDSolveValue[*eqns*, u, {t, t_{min}, t_{max}}, {x, y} ∈ Ω] solves the time-dependent partial differential equations *eqns* over the region Ω.

Week of Monday January xx

Projectile Motion — Part 1

Outline

1- Theory of Projectile Motion

2- Kinematics and Dynamics

3- Ideal projectile motion without air resistance

4- Free fall with air resistance

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EXERCISE 0 (review exercise)
Plot a graph of the function f(x) = exp(3 cos(x)) for –20 < x < 20.
Evaluate f(12.0), accurately.
SKETCH THE GRAPH AND WRITE THE VALUE OF f(12.0) ON THE ANSWER SHEET.
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[1] Theory of Projectile Motion

A mass moving in Earth's gravity near the surface of the Earth is called a projectile.

The motion of a projectile is affected by three forces:

▫ the force of Earth's gravity = $\vec{F}_g = m\,\vec{g}$ (vector $\vec{g}$ is downward)

▫ the buoyancy force = $\vec{F}_b = -\rho_{air}\,V\,\vec{g}$ (Archimedes' law; usually very small)

▫ and the aerodynamic force = $\vec{F}_{air}$ (depends on velocity)

In general the motion is three-dimensional, so we would set up a 3D Cartesian coordinate system (x,y,z). However, if there is no sideward aerodynamic force then the particle will only move on a vertical plane, which we'll take to be the xy-plane; x = horizontal coordinate, y = vertical coordinate.

EXERCISE 1 - does not use Mathematica

Sketch a figure that shows...

- the xy-coordinate axes, x toward the right, y upward;
- the projectile at some arbitrary time t, and the position vector $\vec{r}(t)$;
- label x, y, and $\vec{r}$ on the figure.

SKETCH THE FIGURE ON THE ANSWER SHEET.

EXERCISE 2 - does not use Mathematica

Sketch a figure that shows a typical trajectory curve for a projectile moving in the xy-plane.

SKETCH THE FIGURE ON THE ANSWER SHEET.

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[2] Kinematics and Dynamics for Projectile Motion

The equation of motion, in vector form, for the particle motion is

$$m\,\frac{d^2\vec{r}}{dt^2} = \vec{F}_g + \vec{F}_b + \vec{F}_{air}$$

Or, the equations of motion, in coordinate form, are

$$m\,\frac{d^2x}{dt^2} = F_{ax} \qquad \text{and} \quad m\,\frac{d^2y}{dt^2} = -\,m\,g + F_{by} + F_{ay}$$

Note that these are coupled second-order differential equations.
To have a definite solution we will need four initial conditions,

$$x(0) = x_0 \;\;,\;\; y(0) = y_0 \;\;,\;\; v_x(0) = v_{ox} \;\;,\;\; v_y(0) = v_{oy}$$

where

$$v_x(t) = \frac{dx}{dt} \qquad \text{and} \qquad v_y(t) = \frac{dy}{dt}\,.$$

[3] Ideal Projectile Motion Without Air Resistance

If we neglect the effects of the air, then the only force on the projectile is the force of gravity. Of

Near the surface of the Earth there is always buoyancy (usually very small) and an aerodynamic force of air resistance, because of the atmosphere the atmosphere. But these forces are small and we'll start by assuming that they can be neglected. (Later we'll see that neglecting the effects of the air may not be a good approximation in some interesting cases.)

In other words, we'll start by doing calculations with $\vec{F}_{air} = 0$, even though that might not be a very accurate approximation. Later we'll put in the air resistance, and try to determine whether the approximation $\vec{F}_a = 0$ is reasonable.

So, in this section we'll say that the only force on the projectile is gravity,

$$\vec{F} = -mg\,\hat{y} \qquad \text{(near the surface of the Earth),}$$
$$g = 9.81\ m/s^2\ .$$

Force is a vector, so we need to use correct vector notations.

Vector notations

In physics it is important to use correct vector notations.

- A vector is a quantity with both direction and magnitude. The opposite of a vector is a scalar, which has magnitude and sign, but no direction. When you hand-write a vector, draw an arrow over the symbol. In math books, vectors are written in bold face, and scalars in normal face. But handwritten vectors must have an arrow over the symbol.

- Note that $\vec{A}$ denotes a vector; but A does not denote a vector !

- Using Cartesian coordinates, a vector can be expanded in components, like this:

$$\vec{A} = A_1\,\hat{x} + A_2\,\hat{y} + A_3\,\hat{z}\ ,$$

where A_1 and A_2 and A_3 are numbers (possibly including units) or variables.
The other quantities ($\hat{x}$ and $\hat{y}$ and $\hat{z}$) are unit vectors—the direction vectors of the Cartesian axes. When writing a unit vector by hand, it is necessary to put a hat (^) over the symbol.

- Another notation for vectors, which is often more convenient, is

$$\vec{A} = (A_1, A_2, A_3)\ ,$$

where again A_1, A_2 and A_3 are the Cartesian components of $\vec{A}$.

You will need to use correct notations for vectors, and your professors should take off points on homework problems or exams if you use incorrect notations.

The equation of motion is Newton's second law.

Neglecting air resistance, the equation of motion for the projectile is, in vector form,

$$m \frac{d^2 \vec{r}}{dt^2} = -m g \hat{y} ; \qquad \text{or,} \qquad \frac{d^2 \vec{r}}{dt^2} = -g \hat{y} ;$$

Note that the mass of the projectile cancels, so projectiles with equal initial conditions follow the same trajectory.

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Exercise.
Neglecting air resistance, which falls faster — a nail or a hammer?
Answer on the answer sheet.
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In terms of the Cartesian coordinates, x and y, of the particle, $\vec{r} = x \hat{x} + y \hat{y}$;
then the equations of motion are

$$\frac{d^2 x}{dt^2} = 0 \qquad \text{and} \qquad \frac{d^2 y}{dt^2} = -g .$$

The solutions are familiar to you from PHY 183,

$x(t) = x_0 + v_{ox}\, t$ and $y(t) = y_0 + v_{oy}\, t - \frac{1}{2} g t^2 \quad (\star)$

where (x_0 , y_0) is the initial position vector, and (v_{ox} , v_{oy}) is the initial velocity vector.

The equations $(\star)$ apply an ideal projectile. This is an example where the acceleration is constant; $\vec{a}(t) = -g \hat{y}$.

You must never use these equations if the acceleration is not constant !!!

[4] A Projectile Exercise, neglecting the effects of air

WRITE YOUR ANSWERS TO PARTS (A) — (E) ON THE ANSWER SHEET. NEATNESS COUNTS. SLOPPINESS COUNTS NEGATIVELY.

<u>EXERCISE 3. Baseball home run</u>

A baseball batter hits a home run.

As the ball leaves the bat, the coordinates are (x_0, y_0) and the components of the velocity vector are (v_{ox}, v_{oy}). The initial direction of the velocity vector is at angle θ_0 above the horizontal.

(A) Sketch a picture showing the initial conditions for the baseball; label these quantities on the picture: x_0 , y_0 , v_{ox} , v_{oy} , θ_0 .

(B) Assume these parameter values:

$x_0 = 0$; $y_0 = 3$ ft ; $v_0 = 140$ ft/sec ; $\theta_0 = 45$ degrees $= \pi/4$ radians.

[Use feet and seconds for the units. Then you must use the value of g in ft/sec^2! So start by converting g from 9.81 m/s^2 to xx.xx ft/s^2 . Remember, 1 inch = 2.54 cm.]

Use Mathematica to plot the trajectory of the baseball in the xy-plane. Use the command **ParametricPlot.**

If you need help using ParametricPlot, then type ?ParametricPlot (Shift-Enter).
The axes should show distances in feet, so label the axes like this : "x [ft]" and "y[ft]";
that is, put this Option in the ParametricPlot command:

AxesLabel -> { "x [ft]" , "y [ft]" }

Make sure you use appropriate ranges for the x and y axes; that is, put this Option in the ParametricPlot command :

PlotRange -> {{##, ##}, {##, ##}} ;

you supply appropriate values for the ranges.

(C) Use graphical analysis to determine the time t_{grd} when the ball will hit the ground.

(D) Also determine the horizontal distance x_{grd} where the ball will hit the ground.

(E) Based on what you know about home runs, are the results of the calculations physically reasonable? In other words, do you think that the effect of air resistance is negligible? Explain.

In Lesson 4 (next week) we' ll redo the calculation including air resistance.

[6] Free Fall with Air Resistance

WRITE YOUR ANSWERS TO PARTS (A) — (F) ON THE ANSWER SHEET. NEATNESS COUNTS. SLOPPINESS COUNTS NEGATIVELY.

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EXERCISE 4. The falling basketball
Dropping a basketball from a height of 20 meters.
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Introduction.

(1) If there is no wind, then the basketball will fall straight down.

(2) The force of air resistance depends on the speed, size and surface roughness of the ball.

(3) A pretty accurate model for the force of air resistance is that the magnitude of the force is

$$F_{air} = 0.25\, \rho_{air}\, A\, v^2$$

where ρ_{air} = the density of air = 1.225 kg / m^3 ,

and A = the cross-sectional area of the sphere = πR^2 ; R = the radius;

the direction of the force is opposite to the velocity vector.

(4) The gravitational force is $\vec{F}_g = -m\, g\, \hat{y}$; $g = 9.81\ m/s^2$.

(5) The buoyancy force is $\vec{F}_b = +\frac{4}{3}\pi R^3\, \rho_{air}\, g\, \hat{y}$.

(A) Assume the sphere is dropped from rest. That is, the initial velocity is $\vec{v_0}$ = (0, 0). The sphere falls straight down so the position vector at time t is (0, y(t)). Write the equation of motion for the coordinate y(t). Ask someone to check that your equation is correct. WRITE THE ANSWER ON THE ANSWER SHEET.

Now specify these parameter values:

- y_0 = the height of the tower = 20 m
- The initial velocity $v_y(0) = 0$; i.e., the sphere is dropped from rest.
- The sphere is a basketball; so mass = m = 0.625 kg and radius = R = 22.86 cm .

Then type into Mathematica the list of equations, and call the list of equations **eqs**.
The Mathematica command should have this syntax:

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eqs = {the diff. eq., the first initial condition,
  the second initial condition}
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In[ ]:=
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eqs = {m * y''[t] == -m * g + mA * g + 0.25 * rho * A * y'[t]^2,
  y[0] == y0, y'[0] == 0}
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Out[ ]= $\{m\,y''[t] == -g\,m + g\,mA + 0.25\,A\,rho\,y'[t]^2,\ y[0] == y0,\ y'[0] == 0\}$

Hint: In Mathematica an equation must be written with a double equals sign. So, for example, if the equation is A(x) = B(x) then we write the equation in Mathematica as `A[x] == B[x]`.

(B) Use **NDSolveValue** to solve the equations of motion, numerically.

In[ ]:=
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? NDSolveValue
```

Out[ ]=

Symbol

NDSolveValue[*eqns*, *expr*, {x, x_{min}, x_{max}}] gives the value of *expr* with functions determined by a numerical solution to the ordinary differential equations *eqns* with the independent variable x in the range x_{min} to x_{max}.

NDSolveValue[*eqns*, *expr*, {x, x_{min}, x_{max}}, {y, y_{min}, y_{max}}] solves the partial differential equations *eqns* over a rectangular region.

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NDSolveValue[*eqns*, u, {t, t_{min}, t_{max}}, {x, y} ∈ Ω] solves the time-dependent partial differential equations *eqns* over the region Ω.

Because this is the numerical solution of a differential equation, to result will be an **InterpolatingFunction .**

Instructions for using an interpolating function.

(C) Plot the solution, i.e., the height **y(t)** in meters versus the time **t** in seconds.
Label the axes and use appropriate plot ranges.
SKETCH THE GRAPH ACCURATELY ON THE ANSWER SHEET. LABEL THE AXES. NEATNESS COUNTS!

(D) Determine the time **τ** when the ball will hit the ground.
Calculate **τ** accurately!
Hint: Use the command **FindRoot**.
WRITE THE ANSWER ON THE ANSWER SHEET. ACCURACY COUNTS.

(E) Compare the result in (D) to the time to fall 20 meters in vacuum.
WRITE THE ANSWER ON THE ANSWER SHEET. ACCURACY COUNTS.

(F) Galileo famously stated that the time to fall from a high tower is approximately the same for all objects. Discuss this statement in the

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light of your calculations.
WRITE THE ANSWER ON THE ANSWER SHEET.
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Extra Credit on the falling basketball exercise

(EXTRA CREDIT)

How large is the effect of buoyancy?

Calculate the time to fall if you set buoyancy to 0, and compare the result to Exercise 4(D).