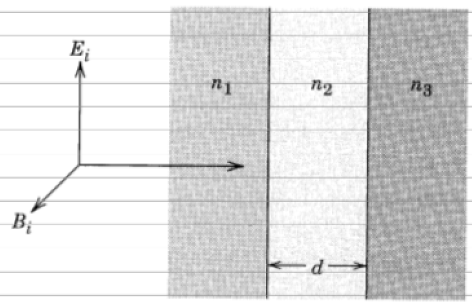
# Problem 7 - 1 .        (Jackson Problem 7.2)

```
Import["jp72.png", "PNG"]
```

**7.2** A plane wave is incident on a layered interface as shown in the figure. The indices of refraction of the three nonpermeable media are $n_1$, $n_2$, $n_3$. The thickness of the intermediate layer is $d$. Each of the other media is semi-infinite.

**(a)** Calculate the transmission and reflection coefficients (ratios of transmitted and reflected Poynting's flux to the incident flux), and sketch their behavior as a function of frequency for $n_1 = 1, n_2 = 2, n_3 = 3$; $n_1 = 3, n_2 = 2, n_3 = 1$; and $n_1 = 2, n_2 = 4, n_3 = 1$.




**Problem 7.2**

This problem will involve a lot of algebra, so I'll solve it using Mathematica.

**FIELDS for normal incidence**

**E1a** = ex $\alpha 1$ exp[$i$ ($k_1 z - \omega$t)] and **H1a** = ey ($n_1 / \mu_0 c$) $\alpha 1$ exp[$i$ ($k_1$z – $\omega$t)]

**E1b** = ex $\alpha 2$ exp[$i$ ( $-k_1 z - \omega$t)] and **H1b** = ey ($-n_1 / \mu_0 c$) $\alpha 2$ exp[$i$ ($-k_1$z – $\omega$t)]

**E2a** = ex $\beta 1$ exp[$i$ ($k_2 z - \omega$t)] and **H2a** = ey ($n_2 / \mu_0 c$) $\beta 1$ exp[$i$ ($k_2$z – $\omega$t)]

**E2b** = ex $\beta 2$ exp[$i$ ($-k_2 z - \omega$t)] and **H2b** = ey ($-n_2 / \mu_0 c$) $\beta 2$ exp[$i$ ($-k_2$z – $\omega$t)]

**E3a** = ex $\gamma$ exp[$i$ ($k_3 z - \omega$t)] and **H3a** = ey ($n_3 / \mu_0 c$) $\gamma$ exp[$i$ ($k_3$z – $\omega$t)]

```
(* Solve the BOUNDARY CONDITIONS *)
Remove["Global`*"]
bcs =
  {α1 + α2 == β1 + β2,
   n1 * (α1 - α2) == n2 * (β1 - β2),
   β1 * Exp[I * k2 * d] + β2 * Exp[-I * k2 * d] == γ * Exp[I * k3 * d],
   n2 * (β1 * Exp[I * k2 * d] - β2 * Exp[-I * k2 * d]) == n3 * γ * Exp[I * k3 * d]};
solutions = Solve[bcs, {α2, β1, β2, γ}]
```

$$\left\{\left\{\alpha 2 \to \frac{\left(\text{n1 n2} + e^{2 i d k2}\,\text{n1 n2} - \text{n2}^2 + e^{2 i d k2}\,\text{n2}^2 + \text{n1 n3} - e^{2 i d k2}\,\text{n1 n3} - \text{n2 n3} - e^{2 i d k2}\,\text{n2 n3}\right)\alpha 1}{\text{n1 n2} + e^{2 i d k2}\,\text{n1 n2} + \text{n2}^2 - e^{2 i d k2}\,\text{n2}^2 + \text{n1 n3} - e^{2 i d k2}\,\text{n1 n3} + \text{n2 n3} + e^{2 i d k2}\,\text{n2 n3}},\right.\right.$$

$$\beta 1 \to -\frac{2\,\text{n1}\,(\text{n2} + \text{n3})\,\alpha 1}{-\text{n1 n2} - e^{2 i d k2}\,\text{n1 n2} - \text{n2}^2 + e^{2 i d k2}\,\text{n2}^2 - \text{n1 n3} + e^{2 i d k2}\,\text{n1 n3} - \text{n2 n3} - e^{2 i d k2}\,\text{n2 n3}},$$

$$\beta 2 \to \frac{2\,e^{2 i d k2}\,\text{n1}\,(\text{n2} - \text{n3})\,\alpha 1}{\text{n1 n2} + e^{2 i d k2}\,\text{n1 n2} + \text{n2}^2 - e^{2 i d k2}\,\text{n2}^2 + \text{n1 n3} - e^{2 i d k2}\,\text{n1 n3} + \text{n2 n3} + e^{2 i d k2}\,\text{n2 n3}},$$

$$\left.\left.\gamma \to -\frac{4\,e^{i d k2 - i d k3}\,\text{n1 n2}\,\alpha 1}{-\text{n1 n2} - e^{2 i d k2}\,\text{n1 n2} - \text{n2}^2 + e^{2 i d k2}\,\text{n2}^2 - \text{n1 n3} + e^{2 i d k2}\,\text{n1 n3} - \text{n2 n3} - e^{2 i d k2}\,\text{n2 n3}}\right\}\right\}$$

(* calculate the transmission coefficient *)

The transmitted wave is E3a, with amplitude of oscillation = $\gamma$.

The incident wave is E1a, with amplitude of oscillation = $\alpha 1$.

So TC = $\frac{n_3}{n_1}\ \frac{|\gamma|^2}{|\alpha 1|^2}$.

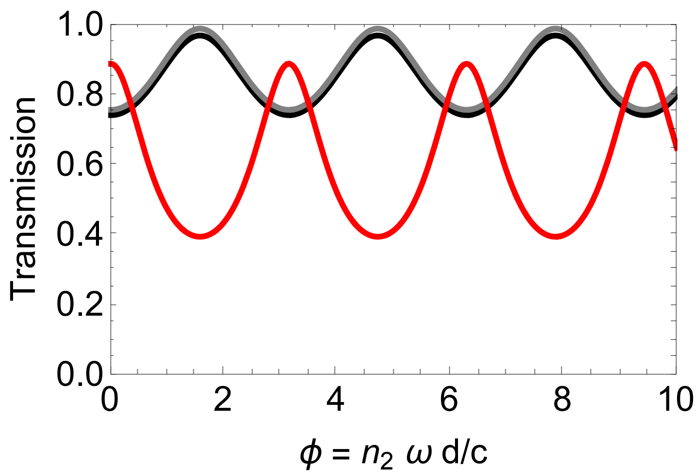
```
Γ = γ /. solutions[[1]] /. {α1 → 1};
num = Numerator[Γ];
den = Denominator[Γ];
numsq = ComplexExpand[num * Conjugate[num]]
densq = ComplexExpand[den * Conjugate[den]];
densq = densq // Simplify;
densq = densq /. {k2 → ϕ / d}; (* this defines a quantity ϕ *)
TC[f_] = (n3 / n1) * (numsq / densq)
```

$16\,\text{n1}^2\,\text{n2}^2$

$$\frac{8\,\text{n1}\,\text{n2}^2\,\text{n3}}{4\,\text{n1}\,\text{n2}^2\,\text{n3} + \text{n1}^2\left(\text{n2}^2 + \text{n3}^2\right) + \text{n2}^2\left(\text{n2}^2 + \text{n3}^2\right) + \left(\text{n1}^2 - \text{n2}^2\right)\left(\text{n2}^2 - \text{n3}^2\right)\text{Cos}[2\,\phi]}$$

Plot the transmission coefficient for {n1,n2,n3} = {1,2,3} and {3,2,1} and {2,4,1},
as a function of $\phi \equiv k_2$ d = $\frac{\omega}{v_2}\ d = \frac{n_2\,\omega\,d}{c}$.

```
ps = {{AbsoluteThickness[3], Black},
   {AbsoluteThickness[3], Gray},
   {AbsoluteThickness[3], Red}};
Plot[
 {0.99 * TC[ϕ] /. {n1 → 1, n2 → 2, n3 → 3},
  1.01 * TC[ϕ] /. {n1 → 3, n2 → 2, n3 → 1 },
  TC[ϕ] /. {n1 → 2, n2 → 4,  n3 → 1 }},
 {ϕ, 0, 10}, Frame → True, FrameLabel → {"ϕ = n₂ ω d/c", "Transmission"},
 BaseStyle → 18, PlotStyle → ps,
 PlotRange → {{0, 10}, {0, 1}}]
```



# Part (b)

Assume $n_1 = n_{lens}$, $n_2 = n_{coating}$, and $n_3 = n_{air} = 1$.
Determine d and $n_2$ such that reflection is 0 for frequency $\omega_0$.

```
TC[ϕ]
```

$$\frac{8\, n1\, n2^2\, n3}{4\, n1\, n2^2\, n3 + n1^2 \left(n2^2 + n3^2\right) + n2^2 \left(n2^2 + n3^2\right) + \left(n1^2 - n2^2\right) \left(n2^2 - n3^2\right) \text{Cos}[2\,\phi]}$$

```
Tcoef = TC[ϕ] /. {n3 → 1, Cos[2 ϕ] → 1 - 2 sϕ^2}
den = Denominator[Tcoef] // Expand // FullSimplify
```

$$\frac{8\, n1\, n2^2}{4\, n1\, n2^2 + n1^2 \left(1 + n2^2\right) + n2^2 \left(1 + n2^2\right) + \left(n1^2 - n2^2\right) \left(-1 + n2^2\right) \left(1 - 2\, s\phi^2\right)}$$

$$2\,(1 + n1)^2\, n2^2 - 2\,(n1 - n2)\,(n1 + n2)\left(-1 + n2^2\right) s\phi^2$$

Transmission Coefficient = N/D

$N = 4\, n_1\, n_2^2$

$D = n_2^2\,(n_1 + 1)^2 + (n_2^2 - n_1^2)(n_2^2 - 1)\sin^2(\phi)$ where $\phi = n_2 \omega d/c$.

## Coating with reflectance = 0

Set N = D so that the transmission coefficient is 1.

$$\therefore\ 0 = n_2^2\,(n_1 - 1)^2 + (n_2^2 - n_1^2)\,(n_2^2 - 1)\sin^2(\phi)$$

We can solve this with $\sin^2(\phi) = 1$ and $n_2 = \sqrt{n_1}$.

Then $\phi = \pi/2 \implies d = \frac{\pi c}{2 n_2 \omega}$