Homework Assignment #10

due Friday Nov 8

1. 8-1. (A) Evaluate $P\int_{-1}^{+1} \frac{dx}{x}$

(B) Evaluate $P\int_{-1}^{+1} \frac{dx}{x} \frac{1}{x+2}$

(A) $\{ \int_{-1}^{-\epsilon} + \int_{\epsilon}^{1} \} \frac{dx}{x} = \ln(\epsilon) + \ln(1/\epsilon) = 0$

(B) $\{ \int_{-1}^{-\epsilon} + \int_{\epsilon}^{1} \} \frac{dx}{x(x+2)} = \{ \int_{-1}^{-\epsilon} + \int_{\epsilon}^{1} \} \frac{dx}{2} \{ \frac{1}{x} - \frac{1}{x+2} \} =$

$= 0 - \frac{1}{2} \{ \ln(3) - \ln(1) \} = -\frac{1}{2} \ln(3)$

(2 points)

In[∘]:=
```
Integrate[1 / x / (x + 2), {x, -1, 1}, PrincipalValue → True]
```

Out[∘]= $-\frac{\text{Log}[3]}{2}$

2. 8-2. Jackson Problem 7.22

In[∘]:=
```
p722
```

Out[∘]=

7.22 Use the Kramers–Kronig relation (7.120) to calculate the real part of $\epsilon(\omega)$, given the imaginary part of $\epsilon(\omega)$ for positive ω as

(a) $\text{Im}\, \epsilon/\epsilon_0 = \lambda[\theta(\omega - \omega_1) - \theta(\omega - \omega_2)], \qquad \omega_2 > \omega_1 > 0$

(b) $\text{Im}\, \epsilon/\epsilon_0 = \dfrac{\lambda\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$

In each case sketch the behavior of $\text{Im}\, \epsilon(\omega)$ and the result for $\text{Re}\, \epsilon(\omega)$ as functions of ω. Comment on the reasons for similarities or differences of your results as compared with the curves in Fig. 7.8. The step function is $\theta(x) = 0$, $x < 0$ and $\theta(x) = 1, x > 0$.

In[∘]:=
```
eq7120
```

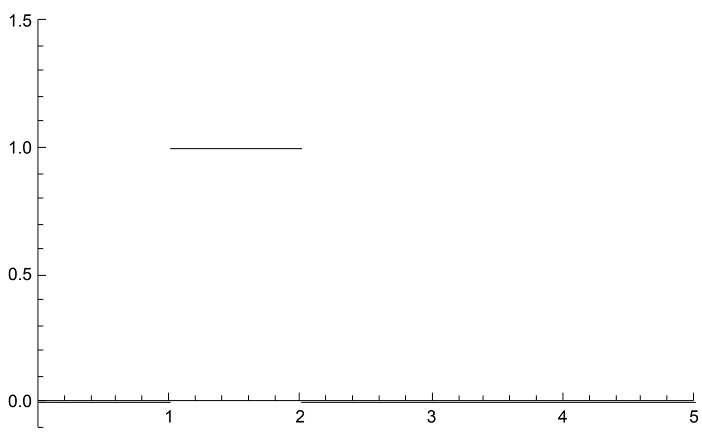
Out[∘]=

$$\text{Re}\, \epsilon(\omega)/\epsilon_0 = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\omega' \,\text{Im}\, \epsilon(\omega')/\epsilon_0}{\omega'^2 - \omega^2}\, d\omega'$$

$$\frac{1}{a^2 - \omega^2} = \frac{1}{2\,a}\left(\frac{1}{a - \omega} + \frac{1}{a + \omega}\right)$$

In[∘]:=
```
(*A*)
θ[x_] = HeavisideTheta[x];
fi[ω_] = θ[ω - 1] - θ[ω - 2];
Plot[fi[x], {x, 0, 5}, PlotRange → {{0, 5}, {-0.1, 1.5}},
 PlotStyle → {{Red, AbsoluteThickness[3]}}, ImageSize → Medium]
```

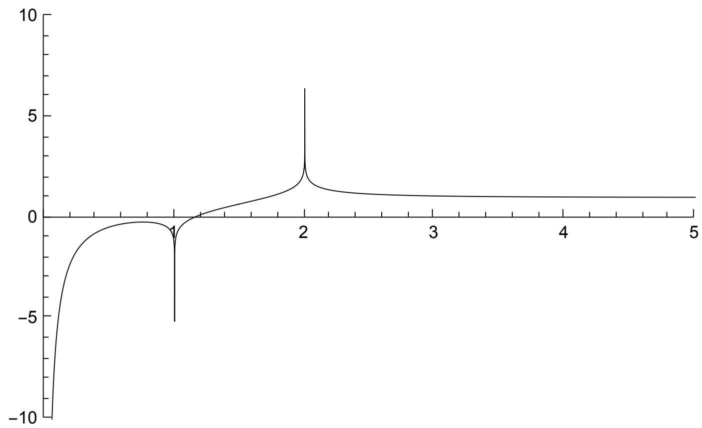
Out[∘]=




```
test11 = Assuming[0 < a < 1, Integrate[x * fi[x] / (a - x), {x, 1, 2}, PrincipalValue → True]];
test12 = Assuming[1 < a < 2, Integrate[x * fi[x] / (a - x), {x, 1, 2}, PrincipalValue → True]];
test13 = Assuming[a > 2, Integrate[x * fi[x] / (a - x), {x, 1, 2}, PrincipalValue → True]];
test2 = Assuming[a > 0, Integrate[x * fi[x] / (a + x), {x, 1, 2}]];
f11 = 1 + (2 / Pi) * 1 / (2 * a) * (test11 - test2) // Simplify;
f12 = 1 + (2 / Pi) * 1 / (2 * a) * (test12 - test2) // Simplify;
f13 = 1 + (2 / Pi) * 1 / (2 * a) * (test13 - test2) // Simplify;
```

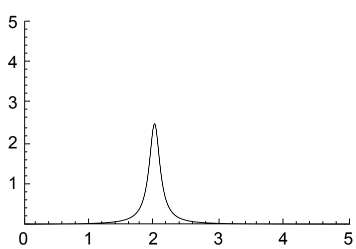
In[∘]:=
```
Show[
 Plot[f11 /. a → ω, {ω, 0, 1}, PlotRange → {{0, 5}, {-10, 10}}],
 Plot[f12 /. a → ω, {ω, 1, 2}, PlotRange → {{0, 5}, {-10, 10}}],
 Plot[f13 /. a → ω, {ω, 2, 5}, PlotRange → {{0, 5}, {-10, 10}}]]
```

Out[∘]=



In[∘]:=
```
(*B*)
Θ[x_] = HeavisideTheta[x];
{ω0, γ, λ} = {2, γ = 2 / 10, λ = 1};
gi[ω_] = λ * γ * ω / ((ω0^2 - ω^2)^2 + γ^2 * ω^2);
Plot[gi[x], {x, 0, 5}, PlotRange → {{0, 5}, {0, 5}}, ImageSize → Small]
```

Out[∘]=



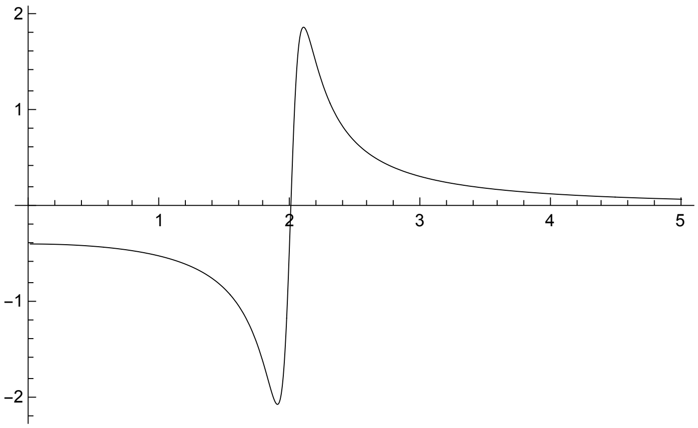

```
Remove[a]
test2 = Assuming[a ∈ Reals && a > 0,
  Integrate[x * gi[x] / (a^2 - x^2), {x, 0, Infinity}, PrincipalValue → True]]
```

Out[∘]=

$$\frac{25\left(-4+a^2\right)\pi}{800-398\,a^2+50\,a^4}$$

In[∘]:= `Plot[test2 /. {a → x}, {x, 0, 5}, PlotRange → All]`

Out[∘]=



(6 points)

3. 8-3. Jackson Problem 7.25

In[∘]:= p725

Out[∘]=

7.25 Equation (7.67) is an expression for the square of the index of refraction for waves propagating along field lines through a plasma in a uniform external magnetic field. Using this as a model for propagation in the magnetosphere, consider the arrival of a whistler signal (actually the Brillouin precursor and subsequently of Section 7.11).

(a) Make a reasonably careful sketch of $c\,dk/d\omega$, where $k = \omega n(\omega)/c$, for the positive helicity wave, assuming $\omega_p/\omega_B \gtrsim 1$. Indicate the interval where $c\,dk/d\omega$ is imaginary, but do not try to sketch it there!

(b) Show that on the interval, $0 < \omega < \omega_B$, the minimum of $c\,dk/d\omega$ occurs at $\omega/\omega_B \simeq \frac{1}{4}$, provided $\omega_p/\omega_B \geq 1$. Find approximate expressions for $c\,dk/d\omega$ for ω near zero and for ω near ω_B.

(c) By means of the method of stationary phase and the general structure of the solution to Problem 7.20a, show that the arrival of a whistler is signaled by a rising and falling frequency as a function of time, the falling frequency component being the source of the name.

In[∘]:= eq766
eq767

Out[∘]=

$$\omega_B = \frac{eB_0}{m} \qquad (7.66)$$

Out[∘]=

$$\epsilon_{\pm}/\epsilon_0 = 1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_B)} \qquad (7.67)$$

part(a)

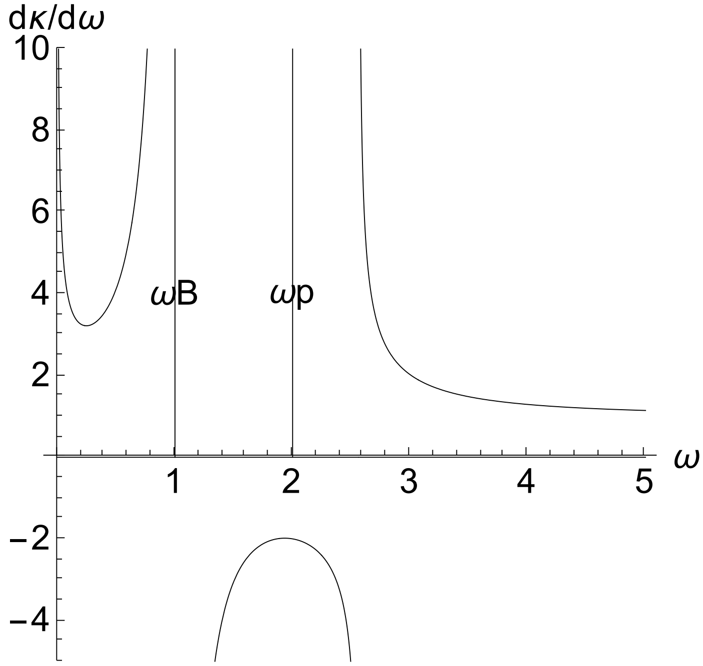
```
In[∘]:= (*A*)
ϵp[ω_] = ϵ0 * (1 - ωp^2 / ω / (ω - ωB)) /. {ϵ0 → 1};
np[ω_] = Sqrt[ϵp[ω]];
κ[ω_] = ω * np[ω] / c /. {c → 1};
fA[ω_] = D[κ[ω], ω] // Simplify
fun[ω_] = fA[ω] /. {ωB → 1, ωp → 2};
```

Out[∘]=

$$\frac{2\,\omega^3 - 4\,\omega^2\,\omega B + 2\,\omega\,\omega B^2 + \omega B\,\omega p^2}{2\,\omega\,(\omega - \omega B)^2\sqrt{1 - \frac{\omega p^2}{\omega^2 - \omega\,\omega B}}}$$

In[∘]:=
```
Show[
 Plot[{Re[fun[ω]], Im[fun[ω]]}, {ω, 0, 5},
  PlotStyle → {Blue, Red}, PlotRange → {All, {-5, 10}},
  AxesLabel → {"ω", "dκ/dω"}, BaseStyle → 18, AspectRatio → 1],
 Graphics[{
   Line[{{1, 0}, {1, 10}}], Text["ωB", {1, 4}, {0, 0}],
   Line[{{2, 0}, {2, 10}}], Text["ωp", {2, 4}, {0, 0}]}]  ]
```

Out[∘]=



part(b) : $\omega \in (0, \omega_B)$

```
(* minimum of fA ; for te case ωp = 2 ωB ...*)
D[fun[ω], ω] // FullSimplify
FindRoot[-1 + 4 * x + x^3 - x^4 == 0, {x, 0.25}]
(* the minimum occurs at ω = 0.247*ωB *)
```

Out[∘]=

$$\frac{4\,\omega\,\left(-1+4\,\omega+\omega^3-\omega^4\right)}{\left((-1+\omega)\,\omega\right)^{5/2}\,\left(-4+(-1+\omega)\,\omega\right)^{3/2}}$$

Out[∘]= $\{x \to 0.247158\}$

In[∘]:=
```
(* limit ω → 0 *)
Series[fA[ω], {ω, 0, 0}] // Normal;
limit1 = % // PowerExpand // Simplify
```

Out[∘]=

$$\frac{\omega p}{2\,\sqrt{\omega}\,\sqrt{\omega B}}$$

In[∘]:=
```
(* limit ω → ωB *)
Series[fA[ω], {ω, ωB, 0}] // Normal;
limit2 = % // Expand // FullSimplify
```

Out[∘]=
$$-\frac{1}{4\,\omega p^4}\,\omega B\left(-\frac{\omega p^2}{(\omega-\omega B)\;\omega B}\right)^{3/2}\left(\omega B^3-3\,\omega B\,\omega p^2+\omega\left(-\omega B^2+\omega p^2\right)\right)$$

part(c) : arrival of a whistler

The question refers to Problem 7.20a, which is shown below:

In[∘]:=
```
ref7201
ref7202
```

Out[∘]=

7.20 A homogeneous, isotropic, nonpermeable dielectric is characterized by an index of refraction $n(\omega)$, which is in general complex in order to describe absorptive processes.

Out[∘]=

(a) Show that the general solution for plane waves in one dimension can be written

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\omega\, e^{-i\omega t}\left[A(\omega)e^{i(\omega/c)n(\omega)x}+B(\omega)e^{-i(\omega/c)n(\omega)x}\right]$$

where $u(x,t)$ is a component of **E** or **B**.

Now, what is the method of stationary phase?

Look it up on Wikipedia (see below)

In[∘]:= msp

Out[∘]=

An example [edit]

Consider a function

$$f(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} F(\omega) e^{i[k(\omega)x - \omega t]}\, d\omega.$$

The phase term in this function, $\phi = k(\omega)\, x - \omega\, t$, is stationary when

$$\frac{d}{d\omega}\left(k(\omega)x - \omega t\right) = 0$$

or equivalently,

$$\frac{dk}{d\omega} = \frac{t}{x}.$$

It just means that the group velocity is dω/dk.

So the time of arrival from distance D goes like

$$\text{time} = \frac{D}{v_{group}} = \left(\frac{dk}{d\omega}\right) D$$

Now look at the graph of dk/dω in part (A).

In some frequency ranges the arrival time increases with frequency; i.e., the higher frequencies arrive later than lower frequencies.

In some frequency ranges, the higher frequencies arrive earlier than the lower frequencies; LIKE A MUSICAL DESCENDING TONE. That's called a whistler.

(6 points)

1. 8-4. Prove equation (7.122).

In[∘]:= eq7122

Out[∘]=

fact, quite general, as shown above (Section 7.10.C). The plasma frequency can therefore be *defined* by means of (7.59) as

$$\omega_p^2 = \lim_{\omega\to\infty}\{\omega^2[1 - \epsilon(\omega)/\epsilon_0]\}$$

Provided the falloff of Im $\epsilon(\omega)$ at high frequencies is given by (7.114), the first Kramers–Kronig relation yields a *sum rule for* ω_p^2:

$$\omega_p^2 = \frac{2}{\pi}\int_0^\infty \omega \,\mathrm{Im}\,\epsilon(\omega)/\epsilon_0 \, d\omega \qquad (7.122)$$

This relation is sometimes known as the sum rule for oscillator strengths. It can be shown to be equivalent to (7.52) for the dielectric constant (7.51), but is obviously more general.

Start with the Kramers and Kronig relations.

In[∘]:= eq7120A

Out[∘]=

$$\mathrm{Re}\,\epsilon(\omega)/\epsilon_0 = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\omega' \,\mathrm{Im}\,\epsilon(\omega')/\epsilon_0}{\omega'^2 - \omega^2}\, d\omega'$$

$$\mathrm{Im}\,\epsilon(\omega)/\epsilon_0 = -\frac{2\omega}{\pi} P \int_0^\infty \frac{[\mathrm{Re}\,\epsilon(\omega')/\epsilon_0 - 1]}{\omega'^2 - \omega^2}\, d\omega' \qquad (7.120)$$

Now

$$\omega^2\,(1 - \mathrm{Re}\,\{\epsilon/\epsilon_0\}) = -\,(2/\pi)\int_0^\infty x\,\mathrm{Im}\,\{\epsilon(x)/\epsilon_0\}\,\frac{\omega^2}{x^2 - \omega^2}\,dx$$

Take the limit $\omega \longrightarrow \infty$ on both side of the equation...

$$\Longrightarrow\quad \omega_p^2 = \frac{2}{\pi}\int_0^\infty x\,\mathrm{Im}\,\{\epsilon(x)/\epsilon_0\}\,dx$$

as claimed.

(2 points)

2. 8-5. Prove the faltung theorem of Fourier integrals. (See the footnote on page 330.)

In[∘]:= faltung

*Equations (7.103) and (7.105) are recognizable as an example of the *faltung* theorem of Fourier integrals: if $A(t)$, $B(t)$, $C(t)$ and $a(\omega)$, $b(\omega)$, $c(\omega)$ are two sets of functions related in pairs by the Fourier inversion formulas (7.104), and

Out[∘]=
$$c(\omega) = a(\omega)b(\omega)$$

then, under suitable restrictions concerning integrability,

$$C(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(t')B(t-t')\,dt'$$

Proof:

$$X = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(t')\,B(t-t')\,dt'$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a(\omega_1)\,e^{i\omega_1 t'}\,d\omega_1$$

$$\square\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} b(\omega_2)\,e^{i\omega_2(t-t')}\,d\omega_2\,dt'$$

Now use the familiar delta function identity,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega_1-\omega_2)t'}\,dt' = \delta(\omega_1-\omega_2);$$

$\Longrightarrow$

$$X = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a(\omega_1)\,b(\omega_1)\,e^{i\omega_1 t}\,d\omega_1$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} c(\omega)\,e^{i\omega t}\,d\omega = C(t)$$

which proves the theorem.

(2 points)

In[ ]:=
```
totalpoints = 2+6+6+2+2
```

Out[ ]= 18