Homework Assignment 12 due Friday November 22

1. 12-1. Consider an ideal cavity resonator, in the form of a right circular cylinder with inner radius R and length d, and flat end faces. For the TM(0,1,0) mode of oscillation, determine Σ and $\vec{K}$. Also, make figures showing the densities.

Use cylindrical coordinates (ρ,ϕ,z) where $\rho \in$ (0,R) and z $\in$ (0,d).

First, the mode numbers are (n,m,p) = (0,1,0).

Second, the resonant frequency is $\omega_{010} = \frac{1}{\sqrt{\mu\epsilon}} \frac{\xi}{R}$ where $\xi = x_{01} = 2.405$.

```
In[]:= BesselJZero[0, 1] // N
Out[]= 2.40483
```

Third, the fields are $H_z = 0$ and

$$E_z = E_0\, J_0(\xi\rho/R)\, e^{-i\omega t}$$

$$H_\phi = -i\sqrt{\epsilon/\mu}\; E_0\, J_1(\xi\rho/R)\, e^{-i\omega t}$$

- (A) the surface charge density Σ(ϕ,z,t) on the surface S at ρ = R;

Let ΔE_{normal} = the discontinuity of E_{normal} at the surface.

The surface charge density = Σ = $\epsilon_0\, \Delta E_{normal}$.

On the surface S, the normal vector is $\hat{\rho}$ and $E_\rho = 0$.

Σ = 0. Nothing to show.

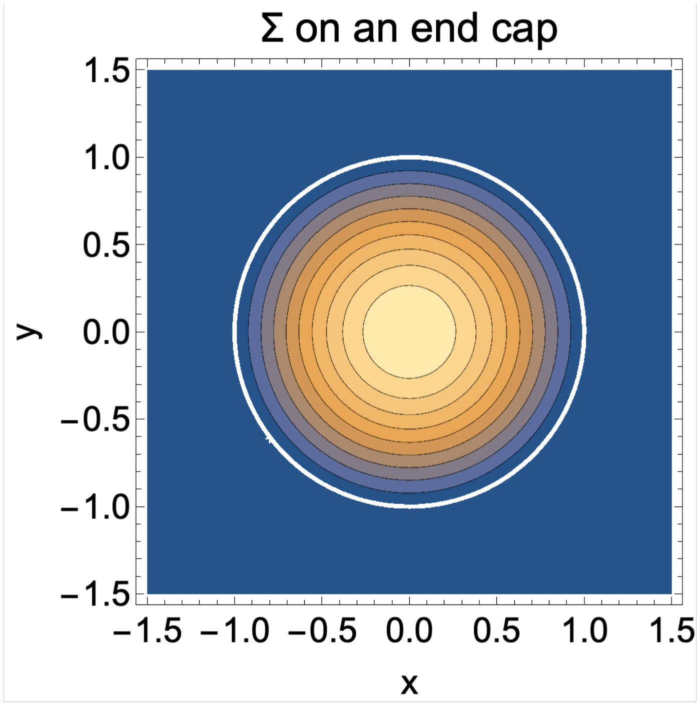
- (B) the surface charge density Σ(ρ,ϕ,t) on the end caps at z = 0 and z = d;

On the end cap at z = d (or 0), the normal vector is $\pm\hat{e}_z$ so

$\Sigma(\rho) = \pm\epsilon_0\, E_0\, J_0(\xi\rho/R)\cos(\omega t)$.

In[∘]:=
```
ContourPlot[
 BesselJ[0, 2.405 * Sqrt[x^2 + y^2]] *
  HeavisideTheta[1 - Sqrt[x^2 + y^2]],
 {x, -1.5, 1.5}, {y, -1.5, 1.5}, AspectRatio → 1, BaseStyle → 18,
 FrameLabel → {"x", "y"},
 PlotLabel → "Σ on an end cap"]
```

Out[∘]=



- (C) the surface current density $\vec{K}(\phi,z,t)$ on the surface S at $\rho = R$;

Let $\Delta\vec{H}_{tangential}$ = the discontinuity of $\vec{H}_{tang.}$ at the surface.

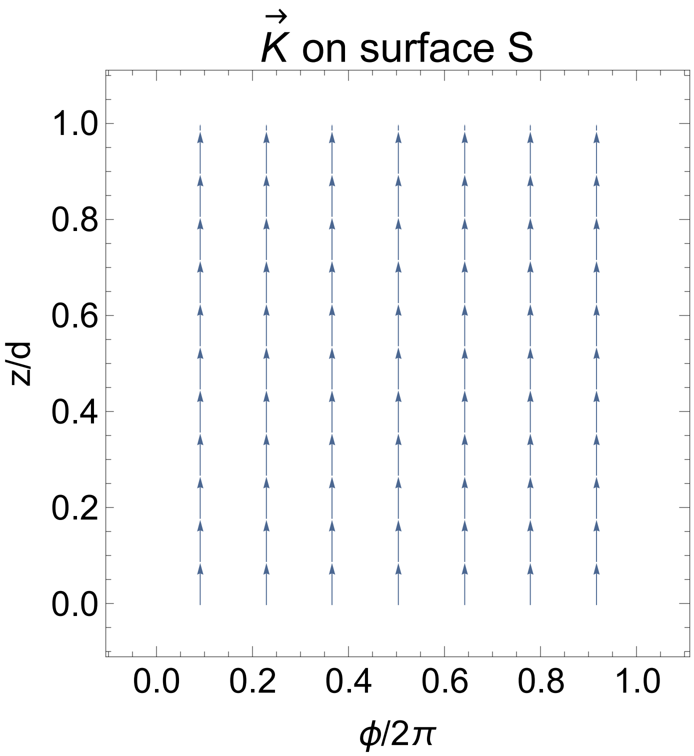
Then $\Delta\vec{H}_{tang.} = \vec{K} \times \hat{n}$ where $\vec{K}$ = the surface current density.

On the surface S, the normal vector is $\hat{\rho}$ and $\vec{H} = \hat{\phi}\, H_\phi$ Thus

$\vec{K}(\phi,z) = \hat{e}_z\, H_\phi = \hat{e}_z \sqrt{\epsilon/\mu}\; E_0\, J_1(\xi) \sin(\omega t)$

In[∘]:=
```
StreamPlot[
 {0, BesselJ[1, 2.405]}, {angle, 0, 1}, {z, 0, 1},
 StreamPoints → Coarse,
 AspectRatio → 1, BaseStyle → 18,
 FrameLabel → {"ϕ/2π", "z/d"}, PlotLabel → "K⃗ on surface S"]
```

Out[∘]=

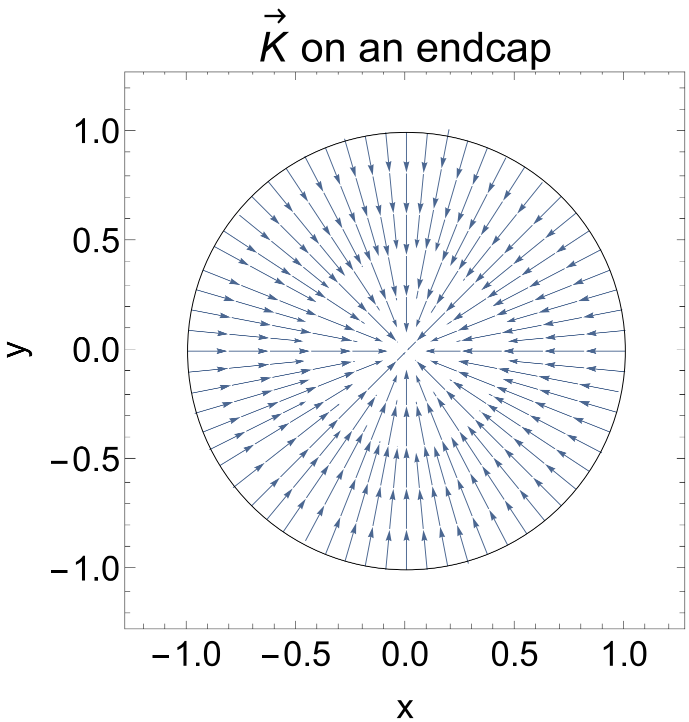


- (D) the surface current density $\vec{K}(\rho,\phi,t)$ on the end caps at z = 0 and z = d.

On the end caps at z = d or 0, the normal vector is $\pm\hat{e}_z$ and $\vec{H} = \hat{\phi}\, H_\phi$ Thus $\vec{K}(\rho,\phi) = \pm\, \hat{e}_\rho\, H_\phi = \pm\, \hat{e}_\rho \sqrt{\epsilon/\mu}\; E_0\; J_1(\xi\rho/R)\, \sin(\omega t)$

In[∘]:=
```
rho = Sqrt[x^2 + y^2];
Show[StreamPlot[
   {-x / rho * BesselJ[1, 2.405 * rho] * HeavisideTheta[1 - rho],
    -y / rho * BesselJ[1, 2.405 * rho] * HeavisideTheta[1 - rho]},
   {x, -1.1, 1.1}, {y, -1.1, 1.1}, AspectRatio → 1,
   BaseStyle → 18, FrameLabel → {"x", "y"}, PlotLabel → "K⃗ on an endcap"],
  Graphics[Circle[{0, 0}, 1]]]
```

Out[∘]=



6 points

2. 12-2. Consider an ideal cavity resonator, in the form of a right circular cylinder with inner radius R and length d, and flat end faces.

For the TE(1,1,1) mode of oscillation, determine Σ and $\vec{K}$. Also, make figures.

The mode numbers are (n,m,p) = (1,1,1).

Second, the resonant frequency is $\omega_{111} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\frac{\xi^2}{R^2} + \frac{p^2\pi^2}{d^2}}$ where $\xi = x'_{11}$ =1.841.

The fields are $E_z = 0$ and

$H_z = \psi\ \sin(\pi z/d)\ e^{-i\omega t}$ where $\psi = H_0\ J_1(\xi\rho/R) \cdot \cos\phi$

$\vec{E}_T = -\frac{i\omega\mu}{\gamma^2}\ \sin(\pi z/d)\ \hat{e}_z \times \nabla_T\psi \quad \cdot e^{-i\omega t}$

$\vec{H}_T = \frac{\pi/d}{\gamma^2}\ \cos(\pi z/d)\ \nabla_T\psi \quad \cdot e^{-i\omega t}$

$\nabla_T\psi = \hat{\rho}\ H_0\ \xi/R \cdot \cos\phi \cdot J_1'(\xi\rho/R) + \hat{\phi}\ H_0\ (-1/\rho) \cdot \sin\phi \cdot J_1(\xi\rho/R)$

In[∘]:=
```
Grad[J1[C * x1] Cos[x2], {x1, x2}, "Polar"]
```

Out[∘]=
$$\left\{C\,\text{Cos}[x2]\,\text{J1}'[C\,x1],\ -\frac{\text{J1}[C\,x1]\,\text{Sin}[x2]}{x1}\right\}$$

- (A) the surface charge density Σ(ϕ,z,t) on the surface S at ρ = R;

The surface charge density = $\Sigma = \epsilon_0\ \Delta E_{normal}$.

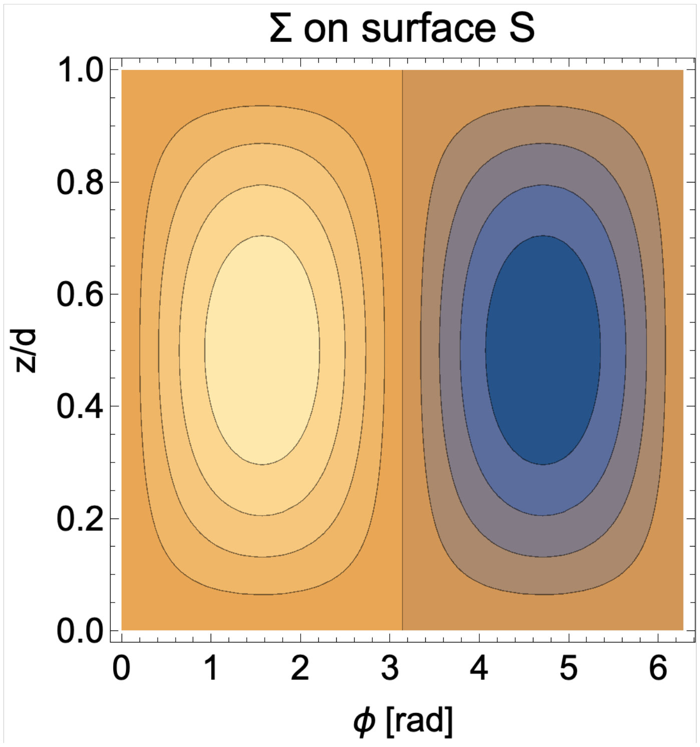
On the surface S, the normal vector is $\hat{\rho}$ and ρ = R;

E_ρ = const · $\sin\phi\ \sin(\pi z/d)\ \cos(\omega t)$

Σ = const · $\sin\phi \cdot \sin(\pi z/d)\ \cos(\omega t)$

In[∘]:=
```
ContourPlot[ Sin[ϕ] * Sin[π * ξ],
 {ϕ, 0, 2 * Pi}, {ξ, 0, 1}, FrameLabel → {"ϕ [rad]", "z/d"},
 AspectRatio → 1, BaseStyle → 18, PlotLabel → "Σ on surface S"]
```

Out[∘]=



- (B) the surface charge density $\Sigma(\rho,\phi,t)$ on the end caps at $z = 0$ and $z = d$;

On the end caps at z = d or 0, the normal vector is $\pm\hat{e}_z$.

$E_z = 0$ so $\Sigma = 0$.   Nothing to show.

- (C) the surface current density $\vec{K}(\phi,z,t)$ on the surface S at $\rho = R$;

Let $\Delta\vec{H}_{tangential}$ = the discontinuity of $\vec{H}_{tang.}$ at the surface.

Then $\Delta\vec{H}_{tang.} = \vec{K} \times \hat{n}$ where $\vec{K}$ = the surface current density.

On the surface S, $\rho = R$; the normal vector is $\hat{\rho}$; and $\vec{H}_{tang.} = \hat{\phi}\, H_\phi + \hat{e}_z\, H_z$

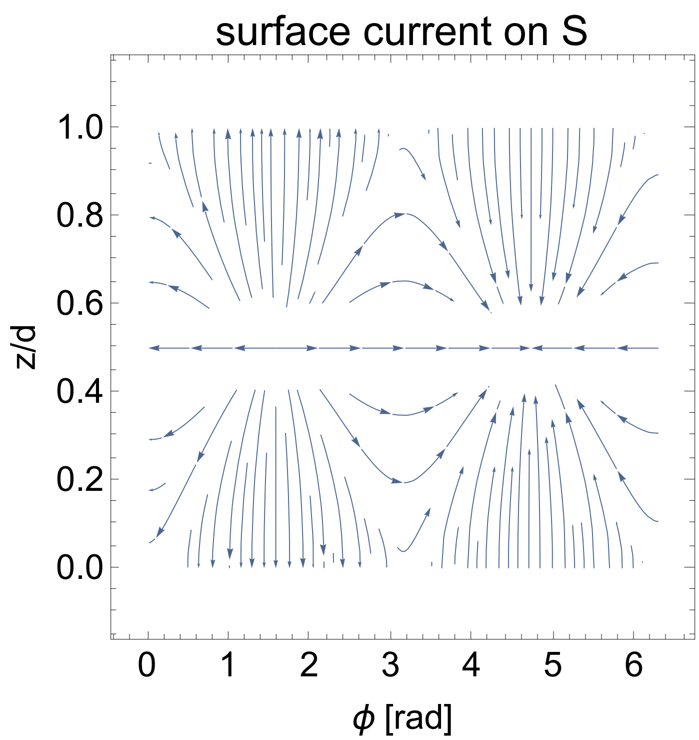
Or, $\vec{K} = \hat{n} \times \vec{H}_{tang} = \hat{e}_z\, H_\phi - \hat{\phi}\, H_z$

Thus $\quad \vec{K}(\phi,z) = -\,\hat{e}_z\, H_0\, J_1\, \frac{\pi}{R d \gamma^2} \sin\phi \cos(\pi z/d) - \hat{\phi}\, H_0\, J_1 \cos\phi \sin(\pi z/d)$

In[∘]:=
```
const = 0.5   (* Pi/(R*d*γ^2) *)
StreamPlot[
 {-Cos[ϕ] * Sin[Pi * ξ], -const * Sin[ϕ] * Cos[Pi * ξ]},
 {ϕ, 0, 2 π}, {ξ, 0, 1},
 AspectRatio → 1, BaseStyle → 18, StreamPoints → Fine,
 FrameLabel → {"ϕ [rad]", "z/d"}, PlotLabel → "surface current on S"]
```

Out[∘]= 0.5

Out[∘]=



- (D) the surface current density $\vec{K}(\rho,\phi,t)$ on the end caps at z = 0 and z = d.

$\Delta\vec{H}_{tang.} = \vec{K} \times \hat{n}$ where $\vec{K}$ = the surface current density.

On the end cap at z = d, the normal vector is $\hat{e}_z$;

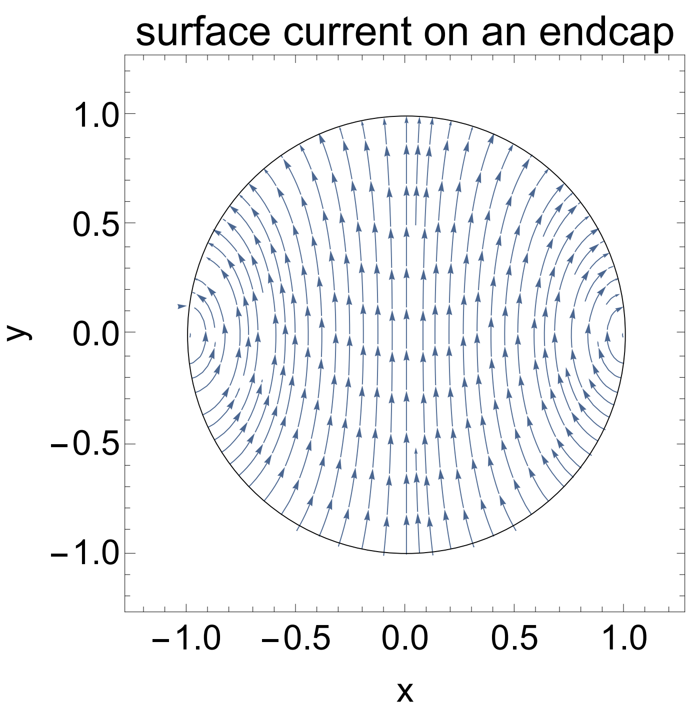
and $\vec{H} = \hat{\rho}\, H_\rho + \hat{\phi}\, H_\phi + \hat{e}_z\, H_z$. Thus

$\vec{K}(\rho,\phi) = \hat{e}_z \times (\hat{\rho}\, H_\rho + \hat{\phi}\, H_\phi) = \hat{\phi}\, H_\rho - \hat{\rho}\, H_\phi$

$= \hat{\rho}\, \sin\phi \cdot (1/\rho) \cdot J_1(\xi\rho/R) + \hat{\phi} \cdot (\xi/R) \cdot \cos\phi \cdot J_1'(\xi\rho/R)$

In[∘]:=
```
rho = Sqrt[x^2 + y^2];
phi = ArcTan[x, y];
J1p[s_] = D[BesselJ[1, s], s];
Kx[x_, y_] = Cos[phi] * Sin[phi] / rho * BesselJ[1, 1.841 * rho] -
   Sin[phi] * Cos[phi] * (1.841) * J1p[1.841 * rho];
Ky[x_, y_] = Sin[phi] * Sin[phi] / rho * BesselJ[1, 1.841 * rho] +
   Cos[phi] * Cos[phi] * (1.841) * J1p[1.841 * rho];
Show[StreamPlot[
  {Kx[x, y] * HeavisideTheta[1 - rho],
   Ky[x, y] * HeavisideTheta[1 - rho]},
  {x, -1.1, 1.1}, {y, -1.1, 1.1}, StreamPoints → Fine,
  AspectRatio → 1, BaseStyle → 18,
  FrameLabel → {"x", "y"}, PlotLabel → "surface current on an endcap"],
 Graphics[Circle[{0, 0}, 1]]]
```

Out[∘]=



6 points

In[∘]:=
```
ArcTan[0.5, 0.2] * 180 / Pi
```
Out[∘]=
```
21.8014
```

3. 12-3. Jackson Problem 8.4. (see Mathematica notebook)

10 points

In[∘]:=
```
jp8p4
```
Out[∘]=

8.4 Transverse electric and magnetic waves are propagated along a hollow, right circular cylinder with inner radius R and conductivity σ.

(a) Find the cutoff frequencies of the various TE and TM modes. Determine numerically the lowest cutoff frequency (the dominant mode) in terms of the tube radius and the ratio of cutoff frequencies of the next four higher modes to that of the dominant mode. For this part assume that the conductivity of the cylinder is infinite.

(b) Calculate the attenuation constants of the waveguide as a function of frequency for the lowest two distinct modes and plot them as a function of frequency.

4. 12-4. Jackson Problem 8.6. (see Mathematica notebook)

5 points

In[∘]:= jp8p6

Out[∘]=

8.6 A resonant cavity of copper consists of a hollow, right circular cylinder of inner radius R and length L, with flat end faces.

(a) Determine the resonant frequencies of the cavity for all types of waves. With $(1/\sqrt{\mu\epsilon}\, R)$ as a unit of frequency, plot the lowest four resonant frequencies of each type as a function of R/L for $0 < R/L < 2$. Does the same mode have the lowest frequency for all R/L?

(b) If $R = 2$ cm, $L = 3$ cm, and the cavity is made of pure copper, what is the numerical value of Q for the lowest resonant mode?

Homework Problem 12-3
(Jackson Problem 8.4)
Part (b)

TE(11)

In[∘]:=
```
Remove["Global`*"]
```

Set R = 1 and $\mu = 1$ and $\epsilon = 1$ and H0=1.
Recall xp11=1.841.

In[∘]:=
```
{R, μ, ϵ, H0} = {1, 1, 1, 1};
```

In[∘]:=
```
γ11 = ξ / R; ω11 = γ11 / Sqrt[μ * ϵ];
ψ[ρ_] = H0 * BesselJ[1, γ11 * ρ];
```

In[∘]:=
```
J = 2 * Pi * Integrate[ψ[ρ] ^2 * ρ, {ρ, 0, 1}]
Jn = J /. {ξ → 1.841}
```

Out[∘]= $\pi \left(\text{BesselJ}[0, \xi]^2 - \frac{2\,\text{BesselJ}[0, \xi]\,\text{BesselJ}[1, \xi]}{\xi} + \text{BesselJ}[1, \xi]^2\right)$

Out[∘]= 0.749815

In[∘]:=
```
temp1 = 1 / 2 / Sqrt[μ * ϵ] * (ω / ω11)^2 * Sqrt[1 - ω11 ^ 2 / ω ^ 2]
temp2 = μ * Jn
P[ω_] = temp1 * temp2 /. {ξ → 1.841}
```

Out[∘]= $\frac{\sqrt{1 - \frac{\xi^2}{\omega^2}}\;\omega^2}{2\,\xi^2}$

Out[∘]= 0.749815

Out[∘]= $0.110616\,\sqrt{1 - \frac{3.38928}{\omega^2}}\;\omega^2$

Also set $\sigma = 1$.
Recall δ = Sqrt[2/(μ*σ*ω)]

In[∘]:=
```
σ = 1;
δ = Sqrt[2 / (μ * σ * ω)]
```

Out[∘]= $\sqrt{2}\sqrt{\frac{1}{\omega}}$

In[∘]:=
```
temp1 = 1 / (2 * σ * δ) * (ω / ω11) ^2 * (2 * Pi * R)
temp2 = (1 / (μ * ϵ * ω11^2) * (1 - ω11^2 / ω^2) * ψ[R]^2 / R^2 + ω11^2 / ω^2 * ψ[R]^2)
Pp[ω_] = temp1 * temp2 /. {ξ → 1.841}
```

Out[∘]= $\frac{\pi}{\sqrt{2}\,\xi^2\left(\frac{1}{\omega}\right)^{5/2}}$

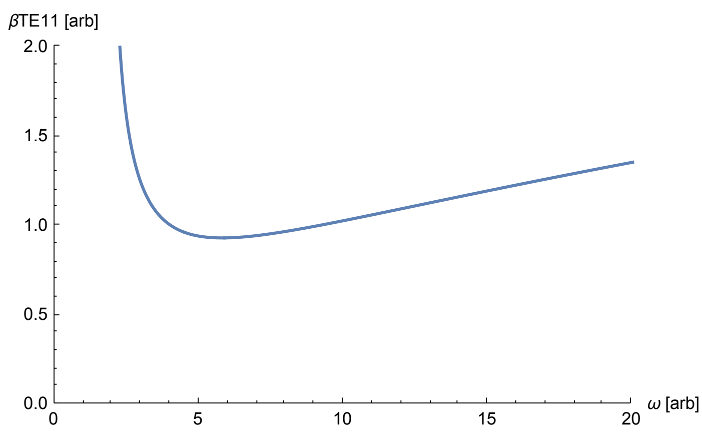
Out[∘]= $\frac{\left(1-\frac{\xi^2}{\omega^2}\right)\text{BesselJ}[1,\xi]^2}{\xi^2}+\frac{\xi^2\,\text{BesselJ}[1,\xi]^2}{\omega^2}$

Out[∘]= $\frac{0.655431\left(0.0998935\left(1-\frac{3.38928}{\omega^2}\right)+\frac{1.1475}{\omega^2}\right)}{\left(\frac{1}{\omega}\right)^{5/2}}$

In[∘]:=
```
βTE11[ω_] = 1 / 2 * Pp[ω] / P[ω]
case1 = Plot[βTE11[ω], {ω, 0, 20}, PlotRange → {{0, 20}, {0, 2}},
  AxesLabel → {"ω [arb]", "βTE11 [arb]"}]
```

Out[∘]= $\frac{2.96265\left(0.0998935\left(1-\frac{3.38928}{\omega^2}\right)+\frac{1.1475}{\omega^2}\right)}{\sqrt{1-\frac{3.38928}{\omega^2}}\sqrt{\frac{1}{\omega}}}$

Out[∘]=



TM(01)

In[∘]:=
```
(* Remove["Global`*"] *)
```

Set R = 1 and μ = 1 and ϵ = 1 and E0=1.

Recall x01 = 2.405.

In[∘]:=
```
{R, μ, ϵ, E0} = {1, 1, 1, 1};
```

In[∘]:=
```
γ01 = ξ / R; ω01 = γ01 / Sqrt[μ * ϵ];
ψ[ρ_] = E0 * BesselJ[0, γ01 * ρ];
```

In[∘]:=
```
J = 2 * Pi * Integrate[ψ[ρ] ^ 2 * ρ, {ρ, 0, 1}]
Jn = J /. {ξ → 2.405}
```

Out[∘]= $\pi \left(\text{BesselJ}[0, \xi]^2 + \text{BesselJ}[1, \xi]^2\right)$

Out[∘]= 0.846581

In[∘]:=
```
temp1 = 1 / 2 / Sqrt[μ * ϵ] * (ω / ω01)^2 * Sqrt[1 - ω01 ^ 2 / ω ^ 2]
temp2 = ϵ * Jn
P[ω_] = temp1 * temp2 /. {ξ → 2.405}
```

Out[∘]= $$\frac{\sqrt{1 - \frac{\xi^2}{\omega^2}}\ \omega^2}{2\ \xi^2}$$

Out[∘]= 0.846581

Out[∘]= $0.0731827 \sqrt{1 - \frac{5.78402}{\omega^2}}\ \omega^2$

Also set $\sigma = 1$ and δ = Sqrt[2 / (σ*μ*ω)]
Recall x01=2.405.

In[∘]:=
```
σ = 1; δ = Sqrt[1 / (σ * μ * ω)]
```

Out[∘]= $$\sqrt{\frac{1}{\omega}}$$

In[∘]:=
```
ψ[ρ]
dnψ[ρ_] = D[ψ[ρ], ρ]
dnψ[R]
```

Out[∘]= BesselJ[0, ξ ρ]

Out[∘]= -ξ BesselJ[1, ξ ρ]

Out[∘]= -ξ BesselJ[1, ξ]

In[∘]:=
```
temp1 = 1 / (2 * σ * δ) * (ω / ω01) ^ 2 * (2 * Pi * R)
temp2 = 1 / (μ ^ 2 * ω01 ^ 2) * dnψ[R] ^ 2
Pp[ω_] = temp1 * temp2  /. {ξ → 2.405}
```

Out[∘]= $\frac{\pi}{\xi^2 \left(\frac{1}{\omega}\right)^{5/2}}$

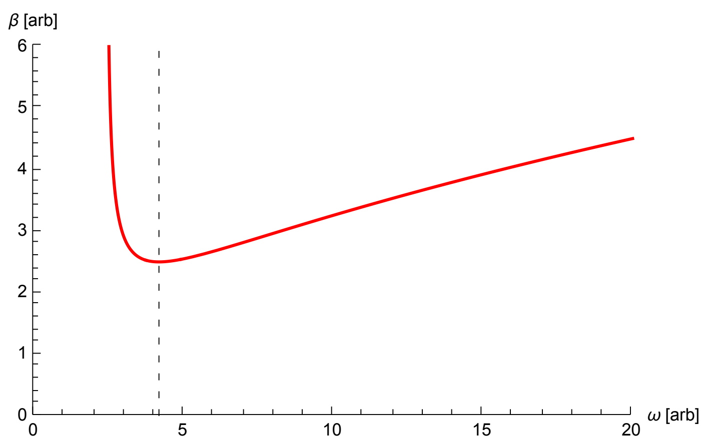
Out[∘]= $\text{BesselJ}[1, \xi]^2$

Out[∘]= $\frac{0.146365}{\left(\frac{1}{\omega}\right)^{5/2}}$

In[∘]:=
```
ωx = Sqrt[3] * ω01 /. {ξ → 2.405}
βTM01[ω_] = 1 / 2 * Pp[ω] / P[ω]
case2 = Plot[βTM01[ω], {ω, 0, 20}, PlotRange → {{0, 20}, {0, 6}},
  AxesLabel → {"ω [arb]", "β [arb]"}, PlotStyle → Red,
  Epilog → {Dashing[{0.01, 0.02}], Line[{{ωx, 0}, {ωx, 6}}]}]
```

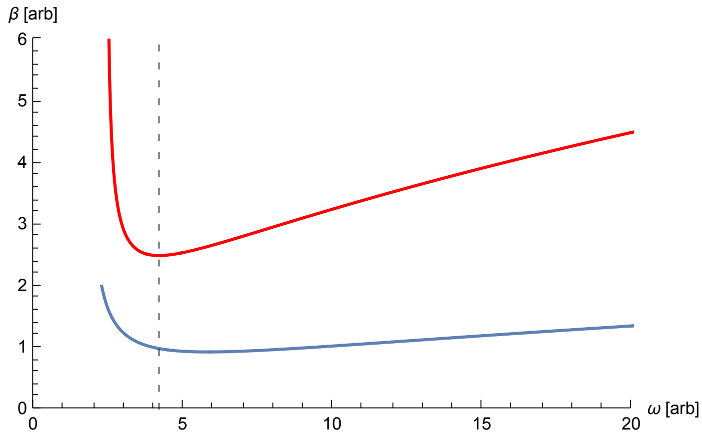
Out[∘]= 4.16558

Out[∘]= $\frac{1.}{\sqrt{1 - \frac{5.78402}{\omega^2}}\ \sqrt{\frac{1}{\omega}}}$

Out[∘]=


In[∘]:=
```
Show[case2, case1]
```

Out[∘]=



In[∘]:=
```
SetDirectory["/Users/OurMacBookAir/Desktop/HOMEWORK/HWSET12"]
FileNames[]
```

Out[∘]= /Users/OurMacBookAir/Desktop/HOMEWORK/HWSET12

Out[∘]= {a1.png, a2.png, a3.png, a4.png, a5.png, a6.png, a7.png, hw12.nb, hw12.solutions copy.nb, hw12.solutions.nb, jackson.soln.8p4.pdf, jackson.soln.8p6.pdf, JP84.solution.nb, JP86.solution.nb, jp8p4.png, jp8p6.png}

In[∘]:=
```
Import["jp8p6.png"]
```

Out[∘]=

8.6 A resonant cavity of copper consists of a hollow, right circular cylinder of inner radius R and length L, with flat end faces.

(a) Determine the resonant frequencies of the cavity for all types of waves. With $(1/\sqrt{\mu\epsilon}\,R)$ as a unit of frequency, plot the lowest four resonant frequencies of each type as a function of R/L for $0 < R/L < 2$. Does the same mode have the lowest frequency for all R/L?

(b) If $R = 2$ cm, $L = 3$ cm, and the cavity is made of pure copper, what is the numerical value of Q for the lowest resonant mode?

Part (a)

This is similar to the wave guide Problem 8.4, except it is a resonant cavity with $\Delta z = L$.

*For **TM modes** the resonant frequencies are*

$$\omega^{(M)}_{mnp} = \frac{1}{\sqrt{\mu\epsilon}} \sqrt{(x_{mn}/R)^2 + (p\pi/L)^2}$$

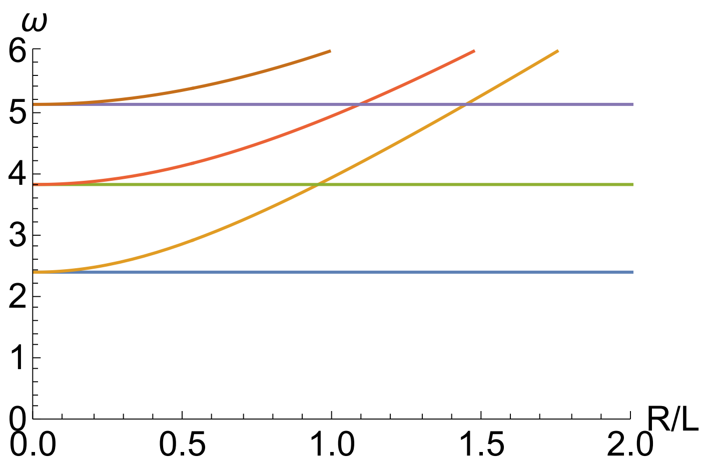
where $J_m(x_{mn}) = 0$ and $p \in \{0, 1, 2, \ldots\}$.

Let $R/L = \beta$;

```
Remove[f1, f2, β]
{zv1, zv2, zv3, zv4} = {2.405, 3.832, 5.136, 5.520}
f1[β_] = Sqrt[(zv1)^2]; f2[β_] = Sqrt[(zv1)^2 + (Pi*β)^2];
f3[β_] = Sqrt[(zv2)^2]; f4[β_] = Sqrt[(zv2)^2 + (Pi*β)^2];
f5[β_] = Sqrt[(zv3)^2]; f6[β_] = Sqrt[(zv3)^2 + (Pi*β)^2];
Plot[{f1[β], f2[β], f3[β], f4[β], f5[β], f6[β]},
 {β, 0, 2}, PlotRange → {{0, 2}, {0, 6}},
 AxesLabel → {"R/L", "ω"}, BaseStyle → 18]
```

Out[•]= {2.405, 3.832, 5.136, 5.52}

Out[•]=



The order of frequencies depends on the value of R/L. Where curves cross, the order of two resonant frequencies changes. For example, for R/L <1 the second mode is (mnp)=(011) but for R/L > 1 the second mode is (110).

---

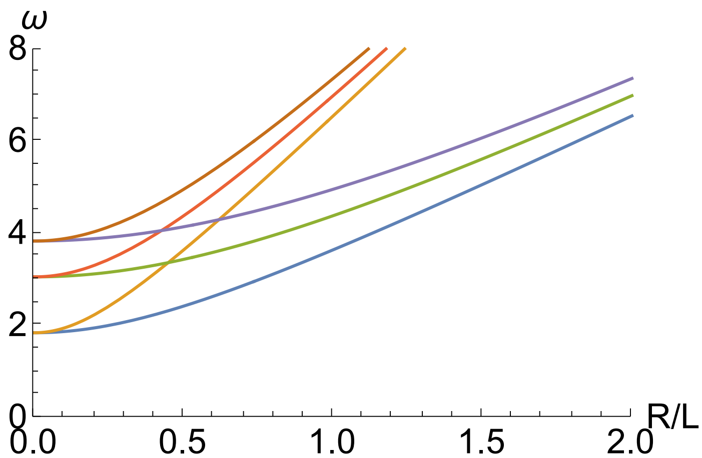
*For **TE modes** the resonant frequencies are*

$$\omega^{(E)}_{mnp} = \frac{1}{\sqrt{\mu\epsilon}} \sqrt{(x'_{mn}/R)^2 + (p\pi/L)^2}$$

where $J'_m(x'_{mn}) = 0$ *and* $p \in \{1, 2, ...\}$.

In[∘]:=
```
Remove[g1, g2, β]
{yv1, yv2, yv3} = {1.841, 3.054, 3.832}
g1[β_] = Sqrt[(yv1)^2 + (Pi*β)^2];
g2[β_] = Sqrt[(yv1)^2 + (2*Pi*β)^2];
g3[β_] = Sqrt[(yv2)^2 + (Pi*β)^2];
g4[β_] = Sqrt[(yv2)^2 + (2*Pi*β)^2];
g5[β_] = Sqrt[(yv3)^2 + (Pi*β)^2];
g6[β_] = Sqrt[(yv3)^2 + (2*Pi*β)^2];
Plot[{g1[β], g2[β], g3[β], g4[β], g5[β], g6[β]},
 {β, 0, 2}, PlotRange → {{0, 2}, {0, 8}},
 AxesLabel → {"R/L", "ω"}, BaseStyle → 18]
```

Out[∘]= {1.841, 3.054, 3.832}

Out[∘]=


Again, the order of modes depends on R/L.

Part (b)

Let R = 2 cm and L = 3 cm.

The lowest resonant mode is TM_{010},

i.e. (mnp)=(010).

Using the results in Section 8.7,

$$Q = \omega \frac{U}{P_{loss}}$$

$$Q = \frac{\mu}{\mu_c} \frac{L}{\delta} \left(1 + \frac{CL}{2A}\right)^{-1} \quad \text{(see equation 8.xx)}$$

where $C = 2\pi R$ and $A = \pi R^2$.

Also, take $\mu = \mu_c = \mu_0$.

$$Q = \frac{L}{\delta} \left(1 + \frac{L}{R}\right)^{-1} = \frac{1.2\text{ cm}}{\delta} = 1.4 \times 10^4 = 14{,}000$$

5 points

Skin depth of copper

```
In[∘]:= (* In SI units *)
{σ, μ} = {6.0*^7, 4.0*^-7 * Pi}
ω = 2.405 * (3.0*^8 / 0.02) (* x01 c/R *)
δ = Sqrt[2 / (σ * μ * ω)]
Q = 0.012 / δ

Out[∘]= {6. × 10^7, 1.25664 × 10^-6}

Out[∘]= 3.6075 × 10^10

Out[∘]= 8.57494 × 10^-7

Out[∘]= 13 994.3
```