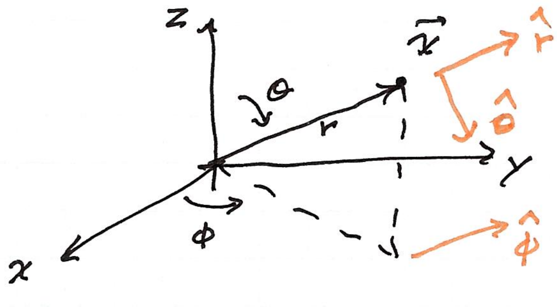
Homework Assignment 13

13-1 Given $\vec{A}(\vec{x}) = -\frac{i\mu_0\omega}{4\pi}\vec{p}\,\frac{e^{ikr}}{r}\,e^{-i\omega t}$

Let $\vec{p} = p\,\hat{e}_z$ and use spherical coordinates (r, θ, ϕ).



(A) The magnetic field is $\nabla\times\vec{A}$. I might as well calculate it in spherical coordinates. (Use Mathematica or look up the curl in spherical coordinates.)

$$B_\phi = -\frac{i\mu_0\omega}{4\pi}\,p\left[\frac{-ik\,e^{ikr}}{r}\sin\theta\right]e^{-i\omega t} + O\left(\frac{1}{r^2}\right)$$

radiation field $\propto \frac{1}{r}$ neglect $O(1/r^2)$

$$\vec{B} = -\frac{\mu_0\omega k}{4\pi}\,p\,\frac{\sin\theta}{r}\,e^{ikr}\,e^{-i\omega t}\,\hat{\phi}$$

Now express the result for $\vec{H} = \vec{B}/\mu_0$.

Note that $\hat{r}\times\vec{p} = p\,\hat{r}\times\hat{e}_z = -p\,\hat{\phi}\sin\theta$

Thus

$$\vec{H} = \frac{\omega k}{4\pi}\,\hat{r}\times\vec{p}\,\frac{e^{i(kr-\omega t)}}{r} = \frac{ck^2}{4\pi}(\hat{r}\times\vec{p})\frac{e^{ikr}}{r}e^{-i\omega t}$$

$(\omega = ck)$ EQUATION 9.19

(B) The electric field

By the Ampère Maxwell equation $\nabla\times\vec{H} = \frac{\partial\vec{D}}{\partial t} = -i\omega\epsilon_0\vec{E}$

$$\vec{E} = \frac{i}{\epsilon_0\omega}\nabla\times\vec{H}$$

Note $\nabla\times\left[\hat{\phi}\,\frac{\sin\theta}{r}\,e^{ikr}\right] = \frac{-ik}{r}\,e^{ikr}\,\hat{\theta}\sin\theta + \underbrace{O(1/r^2)}_{\text{neglect}}$

$$\therefore\ \vec{E} = \frac{i}{\epsilon_0\omega}(-ik)\left(\frac{-\mu_0\omega k}{4\pi}\right)p\,\frac{e^{ikr}}{r}\sin\theta\,\hat{\theta}\;e^{-i\omega t}$$

$$= \frac{-\mu_0 k^2}{4\pi\epsilon_0}\,p\,\frac{e^{ikr}}{r}\sin\theta\,\hat{\theta}\quad e^{-i\omega t}$$

Also, $\vec{H}\times\hat{r} = -\frac{\mu_0\omega k}{4\pi}\;\frac{p\sin\theta}{r}\,e^{ikr}\;\underbrace{\hat{\phi}\times\hat{r}}_{\hat{\theta}}\;e^{-i\omega t}$

Thus $\vec{E} = \frac{k}{\omega\epsilon_0}\vec{H}\times\hat{r} = \frac{\sqrt{\mu_0\epsilon_0}}{\epsilon_0}\vec{H}\times\hat{r} = \sqrt{\frac{\mu_0}{\epsilon_0}}\;\vec{H}\times\hat{r}$

$\vec{E} = Z_0\,\vec{H}\times\hat{r}$ EQUATION 9,19

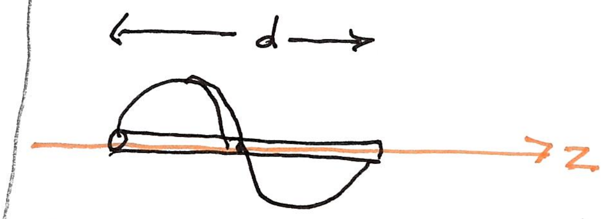
(C) Radiated Power

$$\frac{dP}{d\Omega} = \frac{1}{2}\,\mathrm{Re}\left[r^2\,\hat{r}\cdot(\vec{E}\times\vec{H}^*)\right] = \frac{Z_0 r^2}{2}\,\mathrm{Re}\;\hat{r}\cdot\underbrace{\left[(\vec{H}\times\hat{r})\times\vec{H}^*\right]}_{\hat{r}\,\vec{H}\cdot\vec{H}^*}$$

$$\frac{dP}{d\Omega} = \frac{Z_0 r^2}{2}\left(\frac{\omega k}{4\pi}\right)^2 p^2\,\frac{\sin^2\theta}{r^2}$$

$$= \frac{Z_0\,c^2 k^4}{32\pi^2}\,p^2\sin^2\theta \quad = \frac{Z_0\,c^2}{32\pi^2}\,k^4\,p^2\sin^2\theta$$

EQUATION 9,23



$$I(z,t) = I_0 \sin\left(\frac{2\pi z}{d}\right) e^{-i\omega t}$$

13-2 Jackson Problem 9.16

Linear antenna with $d = \lambda$

In the radiation zone (Eq 9.8)

$$\vec{A}(\vec{x}) = \frac{\mu_0 e^{ikr}}{4\pi r} \int \vec{J}(\vec{x}')\, e^{-ik\hat{n}\cdot\vec{x}'}\, d^3x'$$

$$\vec{J}(\vec{x}') = I(z',t)\,\delta(x')\,\delta(y')\,\hat{e}_z$$

$$= \frac{\mu_0 e^{ikr}}{4\pi r} I_0 \hat{e}_z \int_{-d/2}^{d/2} \sin\left(\frac{2\pi z'}{d}\right) e^{-ik\underbrace{\hat{n}\cdot\hat{e}_z}_{\hat{r}\cdot\hat{e}_z = \cos\theta} z'}\, dz'$$

$$= \frac{\mu_0 e^{ikr}}{4\pi r} I_0 \hat{e}_z \underbrace{\int_{-d/2}^{d/2} \sin kz'\, e^{-ik\cos\theta\, z'}\, dz'}_{= \frac{2i \sin(\pi\cos\theta)}{(-k)\sin^2\theta}}$$

Note: $k = 2\pi/\lambda$ and $d = \lambda$.

$$= \hat{e}_z \frac{(-i)\mu_0 I_0}{2\pi} \frac{e^{ikr}}{kr} \frac{\sin(\pi\cos\theta)}{\sin^2\theta} e^{-i\omega t}$$

- The MAGNETIC FIELD ....

$$\vec{H} = \frac{1}{\mu_0} \nabla\times\vec{A} \xrightarrow[\text{zone}]{\text{radiation}} \frac{1}{\mu_0} ik\,\hat{r}\times\vec{A}$$

$$= -\frac{I_0}{2\pi r} e^{ikr} \frac{\sin[\pi\cos\theta]}{\sin\theta}\,\hat{\phi}$$

- The RADIATED POWER ...

$$\frac{dP}{d\Omega} = \frac{1}{2}\mathrm{Re}\left\{r^2\hat{r}\cdot\left[Z_0(H\times\hat{r})\times\vec{H}^*\right]\right\} = \frac{r^2}{2} Z_0\, \vec{H}\cdot\vec{H}^*$$

$$\frac{dP}{d\Omega} = \frac{r^2}{2} Z_0 \left(\frac{I_0}{2\pi r}\right)^2 \frac{\sin^2[\pi\cos\theta]}{\sin^2\theta}$$

$$\boxed{\frac{dP}{d\Omega} = \frac{Z_0 I_0^2}{8\pi^2} \frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}}$$

(B) Total Power $\quad P = \int_0^\pi \frac{dP}{d\Omega}\, 2\pi \sin\theta\, d\theta$

$$P = \frac{Z_0 I_0^2}{4\pi} \int_{-1}^{1} \frac{du \sin^2(\pi u)}{1-u^2} = \frac{Z_0 I_0^2}{4\pi}\left[\frac{\gamma_E - \cos I(4\pi) + \mathrm{Log}(4\pi)}{2}\right]$$

$$= \frac{Z_0 I_0^2}{4\pi} \times [1.55718\ldots]$$

$\Rightarrow$ RADIATION RESISTANCE, $\quad P = \frac{1}{2} R_{rad} I_0^2$

$$R_{rad} = \frac{Z_0}{2\pi}[1.55718] = 93.4 \text{ ohm}$$

$(Z_0 = 377 \text{ ohm})$

Comparison — Figure 9.1 antenna has

$$R_{rad} = 5(kd)^2 = (197 \text{ ohm})\left(\frac{d}{\lambda}\right)^2$$

Problem 13-4

TE fields in a cylindrical cavity

$$E_z = 0$$

$$H_z = \psi(x,y)\sin\left(\frac{p\pi z}{d}\right) \quad (p = 1\,2\,3 \ldots)$$

$$\vec{E}_T = \frac{-i\omega\mu}{\gamma^2}\sin\left(\frac{p\pi z}{d}\right)\hat{z}\times\nabla_T\psi$$

$$\vec{H}_T = \frac{p\pi}{d\gamma^2}\cos\left(\frac{p\pi z}{d}\right)\nabla_T\psi$$

(factor $e^{-i\omega t}$ and Re Part are understood)

The energy stored in the cavity is

$$U = \int d^3x\left[\frac{\epsilon}{2}E^2 + \frac{\mu}{2}H^2\right]$$

$$U[H_z] = \int da\underbrace{\int_{-d/2}^{d/2}dz}_{d}\,\frac{\mu}{2}\psi^2\underbrace{\sin^2\left(\frac{p\pi z}{d}\right)}_{1/2}\cos^2(\omega t)$$

$$= \frac{\mu d}{4}\int da\,\psi^2\cos^2(\omega t)$$

$$U[E_T] = \int da\cdot d\cdot\frac{\epsilon}{2}\left(\frac{\omega\mu}{\gamma^2}\right)^2\cdot\frac{1}{2}\cdot(\nabla_T\psi)^2\sin^2(\omega t)$$

$$\int(\nabla_T\psi)^2da = -\int\psi\nabla_T^2\psi\,da$$

$$= -\int\psi(-\gamma^2\psi)da = \gamma^2\int\psi^2da$$

$$U[E_T] = \frac{\epsilon d}{4}\frac{\omega^2\mu^2}{\gamma^2}\int da\,\psi^2\sin^2(\omega t)$$

$$U[H_T] = \int da\cdot d\cdot\frac{\mu}{d}\left(\frac{p\pi}{d\gamma^2}\right)^2\cdot\frac{1}{2}\cdot(\nabla_T\psi)^2\cos^2(\omega t)$$

$$= \frac{\mu d}{4}\frac{p^2\pi^2}{d^2\gamma^2}\int da\,\psi^2\cos^2(\omega t)$$

$$U = \int_A \psi^2 \, da \times \{\text{coefficient}\}$$

$$\text{Coefficient} = \frac{\mu d}{4} \cos^2 \omega t$$

$$+ \frac{\epsilon d}{4} \frac{\omega^2 \mu^2}{\gamma^2} \sin^2 \omega t + \frac{\mu d}{4} \frac{p^2 \pi^2}{d^2 \gamma^2} \cos^2 \omega t$$

where $\gamma^2 = \mu \epsilon \omega^2 - \left(\frac{p\pi}{d}\right)^2$

$$\text{coefficient} = \frac{\mu d}{4} \cos^2 \omega t + \frac{\mu d}{4} \left[\frac{\gamma^2 + (p\pi/d)^2}{\gamma^2} \sin^2 \omega t + \frac{(p\pi/d)^2}{\gamma^2} \cos^2 \omega t \right]$$

$$= \frac{\mu d}{4} + \frac{\mu d}{4} \left(\frac{p\pi}{d}\right)^2 \frac{1}{\gamma^2}$$

$$\boxed{U = \frac{\mu d}{4} \left(1 + \left(\frac{p\pi}{d\gamma}\right)^2\right) \int_A \psi^2 \, da}$$

which agrees with EQ. (8,92).

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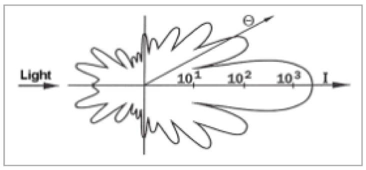
13-4. In one paragraph, what is Mie scattering? Your answer should be complete but concise.

This is the best description I found in a Google search:

Out[∘]=



Mie scattering plot

Mie Scattering

Mie scattering is elastic scattered light of particles that have a diameter similar to or larger than the wavelength of the incident light. The Mie signal is proportional to the square of the particle diameter. Mie scattering is much stronger than Rayleigh scattering and, therefore, a potential source of interference for this weaker light scattering process. There is a strong angular dependency of the scattered intensity especially for smaller particles which has to be considered for successful Mie imaging experiments. Mie scattering is often used to measure flow velocities applying **Particle Image Velocimetry** (**PIV**).

- Mie scattering is the elastic scattering of light by a sphere for which the diameter is comparable to the wavelength of the light.
- Rayleigh scattering is the limit of Mie scattering when $d/\lambda \longrightarrow 0$.
- Applications in atmospheric physics
- Relation to diffraction

13-5. Several research groups in the physics department rely on Cherenkov detectors: HAWC, T2K, IceCube.
For each case, explain how the experiment makes use of Cherenkov radiation. Your answers should be complete but concise.

HAWC = High Altitude Water Cherenkov observatory (Mexico)
observes high energy gamma rays;
charged particles from an air shower strike an array of water Cherenkov detectors.

T2K = Tokai to Kamioka (Japan)
observes neutrinos from an accelerator that interact in the detector;
$\nu_l + A \longrightarrow l^{\pm} + X$; the high energy $l^{\pm}$ radiates light by the Cherenkov process.

IceCube (Antarctica)
observes high energy cosmic neutrinos that interact in the detector; the detector is a volume of ice in Antarctica;
$\nu_l + A \longrightarrow l^{\pm} + X$; the high energy $l^{\pm}$ radiates light by the Cherenkov process.