Grading H. W. A. # 2

2-1) 4 points

2-2) 2 points

2-3) 6 points

2-4) 4 points

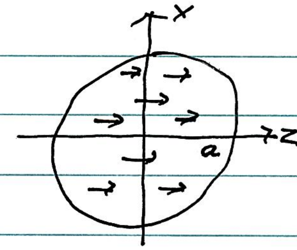
2-5) 4 points

2-6) 2 points

2-7) 2 points

24 points total

Homework Assignment 2



(2-1) Uniformly polarized sphere

Given $\vec{P}(\vec{x}) = P_0\, \hat{e}_z\, \Theta(a-r)$ and $\rho_{free} = 0$.

Determine $\vec{P}(\vec{x})$, $\vec{D}(\vec{x})$, $\vec{E}(\vec{x})$.

Solving field equations $\quad \nabla\times\vec{E} = 0 \quad \Rightarrow \quad \vec{E} = -\nabla\Phi(\vec{x})$

$\vec{D} = \epsilon_0\vec{E} + \vec{P} = -\epsilon_0\nabla\Phi + \vec{P}$ and $\nabla\cdot\vec{D} = 0$

$\therefore\ \nabla^2\Phi = 0$ for both $r<a$ and $r>a$.

Solutions of Laplace's equation $\Phi = \begin{cases} A\, r\cos\theta & \text{for } r<a \\ B\, r\cos\theta + \dfrac{C}{r^2}\cos\theta & \text{for } r>a \end{cases}$

But $B = 0$ because there is no applied field.

Boundary Conditions

- E_θ is continuous at $r=a$ $\Rightarrow$ $A\dfrac{a}{a}\sin\theta = \dfrac{C}{a^3}\sin\theta$

$\boxed{A = C/a^3}$

- D_r is continuous at $r=a$

$\Rightarrow (\epsilon_0 E_r + P_r)_{a-} = (\epsilon_0 E_r + P_r)_{a+}$

$\epsilon_0(-A\cos\theta) + P_0\cos\theta = \epsilon_0\left(+\dfrac{2C}{a^3}\cos\theta\right) \quad \boxed{A = \dfrac{P_0}{\epsilon_0} - \dfrac{2C}{a^3}}$

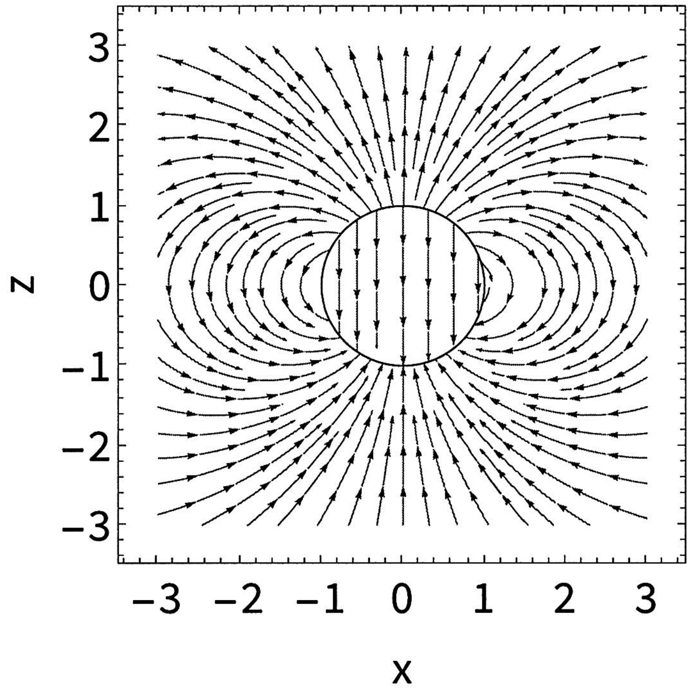
$\therefore$ Thus $A = \dfrac{P_0}{3\epsilon_0}$ and $C = \dfrac{P_0 a^3}{3\epsilon_0}$

$\vec{E}_{in} = -A\hat{e}_z = -\dfrac{P_0\hat{e}_z}{3\epsilon_0}$; $\vec{E}_{out} = -\nabla\left(\dfrac{\vec{p}\cdot\vec{x}}{4\pi\epsilon_0 r^3}\right)$ where $\vec{p} = 4\pi\epsilon_0 C\hat{e}_z = \frac{4}{3}\pi a^3 P_0\hat{e}_z$

$\vec{D}_{in} = \epsilon_0\vec{E}_{in} + \vec{P} = \frac{2}{3}P_0\hat{e}_z$; $\vec{D}_{out} = \epsilon_0\vec{E}_{out} = -\nabla\left(\dfrac{\vec{p}\cdot\vec{x}}{4\pi r^3}\right)$ where $\vec{p} = \frac{4}{3}\pi a^3\vec{P}$

Field lines $\Longrightarrow$

Out[ ]= E field

Out[ ]=


Out[ ]= D field

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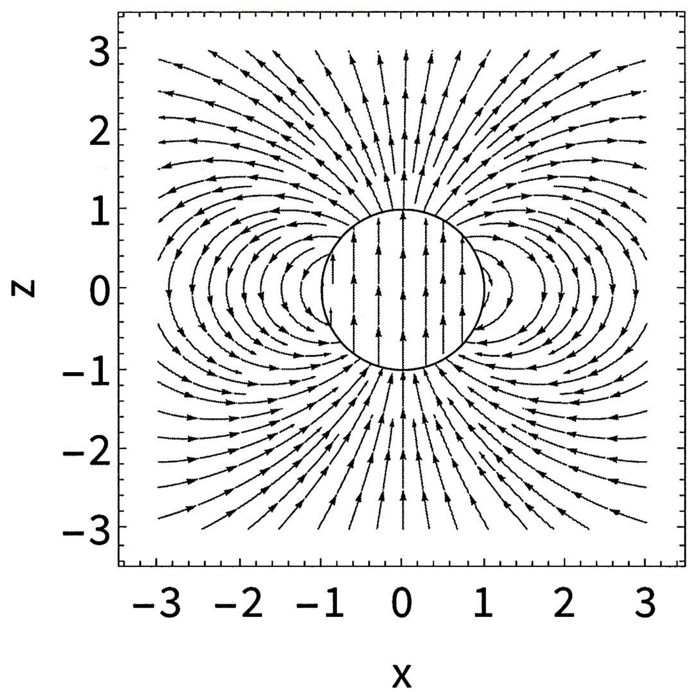
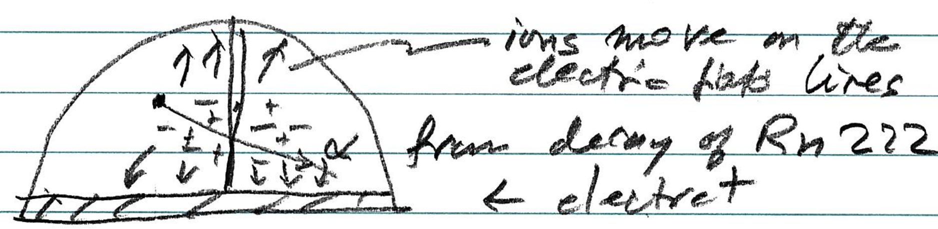


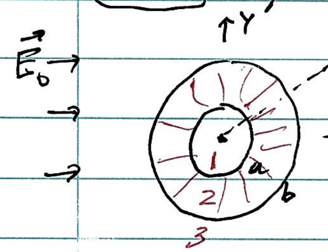
Problem 2-2

an application of electrets

Radon radiation monitor



$\boxed{2\text{-}3}$: Jackson problem 4.8 ; cylindrical shell in an external field



It's a 2D problem ; use (x, y) or (r, ϕ)

Solutions of Laplace's equation are

$$\Phi_1 = C_0\, r\cos\phi \qquad (r<a)$$

Let $\vec{E} = -\nabla\Phi$.

$$\Phi_2 = C_1\, r\cos\phi + \frac{C_2}{r}\cos\phi \qquad (a<r<b)$$

$$\Phi_3 = -E_0\, r\cos\phi + \frac{C_3}{r}\cos\phi \qquad (r>b)$$

Boundary conditions

$E_{tang} \Rightarrow$ $C_0 = C_1 + C_2/a^2$ and $C_1 + C_2/b^2 = -E_0 + C_3/b^2$

$D_{normal} \Rightarrow$ $C_0 = K\left(C_1 - \frac{C_2}{a^2}\right)$ and $K\left(C_1 - \frac{C_2}{b^2}\right) = -E_0 - \frac{C_3}{b^2}$

$$\boxed{K = \epsilon/\epsilon_1}$$

(A) Use Mathematica to solve for C_0, C_1, C_2, C_3.

Then $\vec{E} = -\nabla\Phi = \begin{cases} -C_0\,\hat{e}_x \text{ for } r<a \\ -C_1\,\hat{e}_x - \nabla\left(\frac{C_2 x}{r^2}\right) \text{ for } a<r<b \\ E_0\,\hat{e}_x - \nabla\left(\frac{C_3 x}{r^2}\right) \text{ for } r>b. \end{cases}$

For example, $C_0 = \dfrac{-4b^2 E_0 K}{b^2(K+1)^2 - a^2(K-1)^2}$; C_1, C_2, C_3 similarly.

(check: if $K=1$ then $C_0 = -E_0$ ✓)

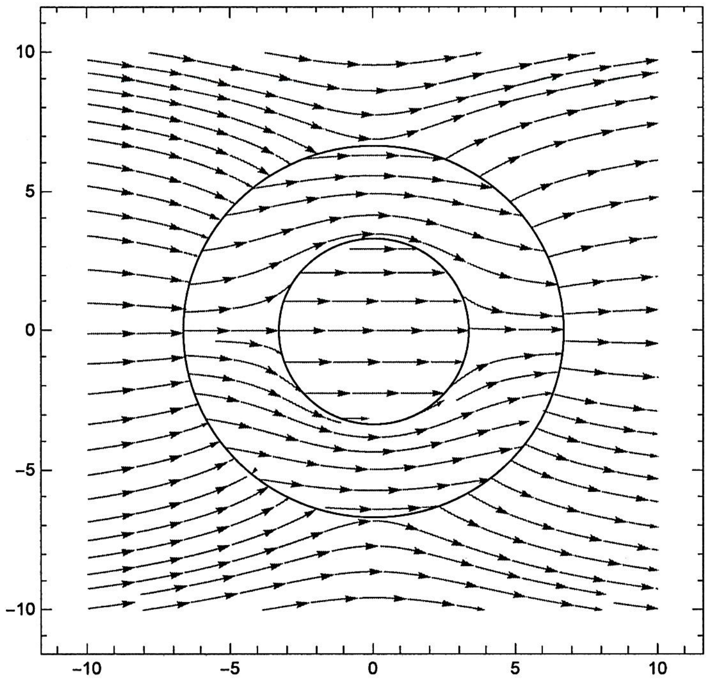
(B) Lines of E field for $b = 2a$ and $K = \frac{\epsilon}{\epsilon_0} = 5$. next page

(C) • solid dielectric cylinder in an applied field ; take the limit $a \to 0$

• cylindrical cavity in a uniform dielectric ; take the limit $b \to \infty$

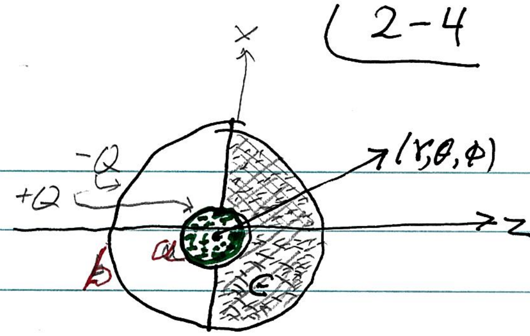
In this case the field in the cavity is $\frac{\pm 4K}{(K+1)^2} E_0\,\hat{e}_x$, which is less than the applied field.

Out[ ]=



2 conducting spheres

Problem 2-4 (Jackson problem 4.10)



(A) $\vec{E}(\vec{x})$ between the spheres

$\nabla\times\vec{E}=0 \Rightarrow$ write $\vec{E}=-\nabla\Phi$.

We can guess $\vec{E}(\vec{x})=\begin{cases}\frac{c_1}{r^2}\hat{r} & \text{for } z<0\\ \frac{c_2}{r^2}\hat{r} & \text{for } z>0\end{cases}$ (radial and $\nabla^2\Phi=0$)

The boundary conditions on $\vec{E}_{\text{tangential}} \Rightarrow c_1=c_2\equiv C$

$\boxed{\vec{E}(\vec{x})=\frac{C}{r^2}\hat{r}}$

Gauss's theorem $\oint D_r r^2 d\Omega = Q$ for $a<r<b$

$$= \frac{\epsilon_0 C}{r^2}\cdot 2\pi r^2 + \frac{\epsilon C}{r^2}\cdot 2\pi r^2 = 2\pi(\epsilon_0+\epsilon)C \Rightarrow \boxed{C=\frac{Q}{2\pi(\epsilon_0+\epsilon)}}$$

(B) Free Surface charge density on the sphere at $r=a$

Recall on the surface of a conductor, $\underbrace{\sigma = D_n(r=a)}_{\text{because } \nabla\cdot\vec{D}=\rho}$ $(\hat{n}=\hat{r})$

$$\therefore \sigma_f = \begin{cases}\frac{\epsilon_0 Q}{2\pi(\epsilon_0+\epsilon)a^2} & \text{for } z<0\\ \frac{\epsilon Q}{2\pi(\epsilon_0+\epsilon)a^2} & \text{for } z>0\end{cases}$$

$D_n\,\delta A=\delta Q$

$D_n=\frac{\delta Q}{\delta A}=\sigma$

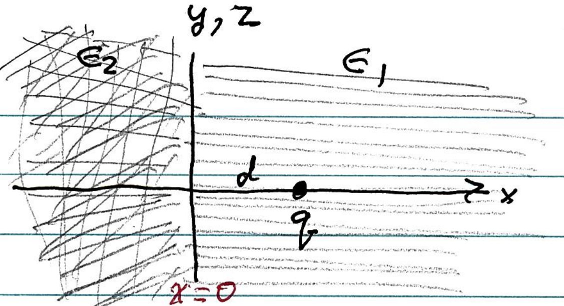
(C) Bound (polarization) surface charge density on the sphere at $r=a$

Recall $\sigma_p=\hat{n}\cdot\vec{P}$ where $\hat{n}=-\hat{r}$ (points out of the body surface)

Also, $\vec{D}=\epsilon_0\vec{E}+\vec{P}=\begin{cases}\epsilon_0\vec{E} & \text{for } z<0\\ \epsilon\vec{E} & \text{for } z>0\end{cases}$; $\therefore \vec{P}=\begin{cases}0 & \text{for } z<0\\ (\epsilon-\epsilon_0)\vec{E} & \text{for } z>0\end{cases}$

$$\text{Thus } \sigma_p = -P_r = \begin{cases}0 & \text{for } z<0\\ \frac{(\epsilon_0-\epsilon)Q}{2\pi(\epsilon_0+\epsilon)a^2} & \text{for } z>0.\end{cases}$$

(2-5)



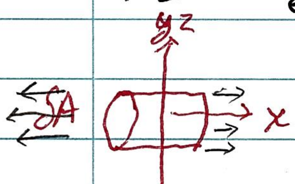
$\vec{d} = (d, 0, 0)$

By the method of images
$$\vec{E}(\vec{x}) = \begin{cases} \dfrac{q}{4\pi\epsilon_1}\dfrac{\vec{x}-\vec{d}}{|\vec{x}-\vec{d}|^3} + \dfrac{q'}{4\pi\epsilon_1}\dfrac{\vec{x}+\vec{d}}{|\vec{x}+\vec{d}|^3} & x>0 \\ \dfrac{q''}{4\pi\epsilon_2}\dfrac{\vec{x}-\vec{d}}{|\vec{x}-\vec{d}|^3} & x<0 \end{cases}$$

where $q' = \dfrac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2} q$ and $q'' = \dfrac{2\epsilon_2}{\epsilon_1+\epsilon_2} q$

(A) The surface charge density There is no free surface charge, so the surface charge density is bound charge.

$$\nabla\cdot\vec{E} = \frac{\rho_{total}}{\epsilon_0} \Rightarrow \{E_{1x}(0^+, y, z) - E_{2x}(0^-, yz)\}\,\delta A = \frac{\delta Q}{\epsilon_0}$$



$$\sigma = \epsilon_0\left[E_{1x} - E_{2x}\right]_{x=0}$$

i.e.,
$$\frac{\sigma_{bound}}{\epsilon_0} = \frac{q}{4\pi\epsilon_1}\frac{(-d)}{R^3} + \frac{q'}{4\pi\epsilon_1}\frac{(d)}{R^3} - \frac{q''}{4\pi\epsilon_2}\frac{(-d)}{R^3}$$

where $R = (d^2+y^2+z^2)^{3/2}$

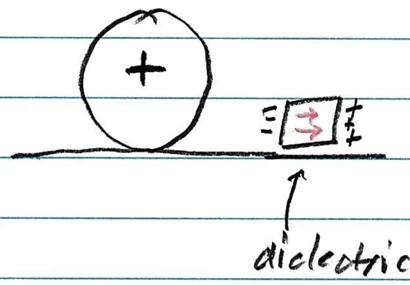
$$= \frac{d}{4\pi R^3} q\left\{\frac{-1}{\epsilon_1} + \frac{\epsilon_1-\epsilon_2}{\epsilon_1(\epsilon_1+\epsilon_2)} + \frac{1}{\epsilon_2}\frac{2\epsilon_2}{\epsilon_1+\epsilon_2}\right\}$$

$$= \frac{qd}{4\pi R^3}\frac{1}{\epsilon_1}\left\{-1 + \frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2} + \frac{2\epsilon_1}{\epsilon_1+\epsilon_2}\right\} = \frac{qd}{4\pi R^3}\frac{-\epsilon_1-\epsilon_2+\epsilon_1-\epsilon_2+2\epsilon_1}{\epsilon_1(\epsilon_1+\epsilon_2)}$$

$$\sigma_{bound} = \frac{qd}{2\pi R^3}\frac{\epsilon_0(\epsilon_1-\epsilon_2)}{\epsilon_1(\epsilon_1+\epsilon_2)}$$

(B) The force on q is attractive if $\epsilon_2 > \epsilon_1$.
or repulsive if $\epsilon_2 < \epsilon_1$

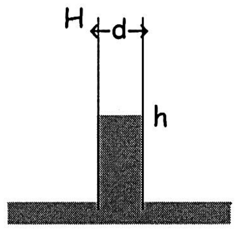
Problem 2-6



negative charge attracted more strongly than positive charge is repelled.

$\Rightarrow$ force $\vec{F}$ toward the sphere

Solution to Problem 2-7



When the plates contact the liquid's surface, an upward force acts on the dielectric liquid. The total charge on each plate remains constant. Energy must be conserved.

- The gravitational energy of the raised liquid is $U_g = \rho\, g\, L\, d\, h^2/2$.
- The initial electrostatic energy of the capacitor is $U_0 = Q^2/(2\,C_0)$

where C_0 = initial capacitance = $\epsilon_0\, HL/d$. Also , $Q/C_0 = V_0 = E_0\, d$.

- When the height of the dielectric liquid in the gap is h, the capacitance is

$$C = [\, \epsilon_0\,(H-h) + \epsilon\, h\,]\,\frac{L}{d} = C_0 + (\epsilon-\epsilon_0)\frac{hL}{d}$$

Then $U_e = \dfrac{Q^2}{2\,C}$ which is less than U_0 .

Energy conservation

$$U_g + U_e = \frac{Q^2}{2\,C_0}$$

$$\rho g\, Ld\,\frac{h^2}{2} = \frac{Q^2}{2}\left(\frac{1}{C_0} - \frac{1}{C}\right) = \frac{Q^2}{2\,C_0\,C}\,(C - C_0)$$

$$\rho g\, Ld\,\frac{h^2}{2} = \frac{Q^2}{2\,C_0\,C}\,(\epsilon - \epsilon_0)\frac{hL}{d}$$

Solve this equation for h.
Use the approximation $C \approx C_0$, valid because $h \ll H$.

$$\frac{E0^2\ (\epsilon - \epsilon 0)}{\rho g}$$

$$h_{\text{equilibrium}} = \frac{E_0^2\,(\epsilon - \epsilon_0)}{\rho g}$$