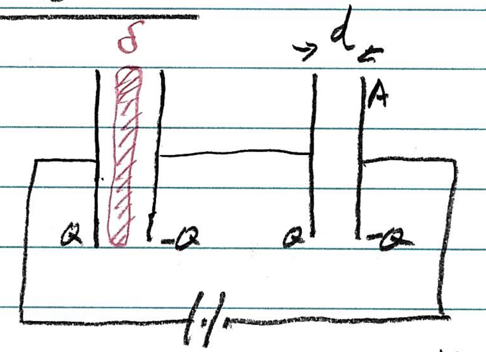
27 points

Homework Assignment #3

Problem 3.1



$V_0 = 50$ volts

$$C_2 = \frac{\epsilon_0 A}{d}$$

$$\frac{1}{C_1} = \frac{\delta}{\epsilon A} + \frac{d-\delta}{\epsilon_0 A}$$

$$\frac{1}{C_{series}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{\delta}{\epsilon A} + \frac{d-\delta}{\epsilon_0 A} + \frac{d}{\epsilon_0 A}$$

$$= \frac{1}{\epsilon_0 A}\left\{2d - \delta + \frac{\delta}{K}\right\} \text{ where } K = \epsilon/\epsilon_0.$$

$$Q = C_{series} V_0$$

$$V_1 = \frac{Q}{C_1} = \frac{C_{series}}{C_1} V_0$$

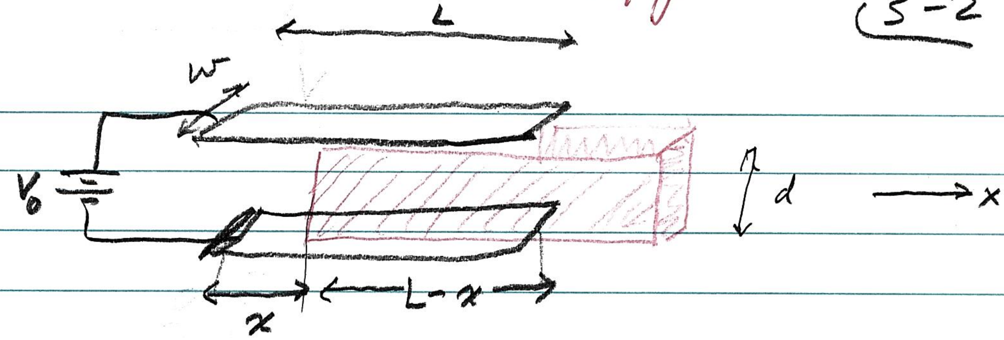
Let $\delta = 0.1\,d$ and $K = 10$

$$\Rightarrow C_1 = \frac{100}{91}\frac{\epsilon_0 A}{d}, \quad C_{series} = \frac{100}{191}\frac{\epsilon_0 A}{d}$$

$$V_1 = \frac{91}{191} V_0 = \boxed{23.82 \text{ volts}}$$

2 points

Problem 3-2



(A) $U = \frac{Q^2}{2C}$ where $C = \frac{\epsilon_0 w x}{d} + \frac{\epsilon w (L-x)}{d}$

$= \frac{\epsilon_0 w}{d}\left[x + K(L-x)\right]$ $\boxed{K = \epsilon/\epsilon_0}$

$$F_x = -\left(\frac{\partial U}{\partial x}\right)_Q = -\left(\frac{-Q^2}{2C^2}\right)\frac{dC}{dx} = +\frac{1}{2}V_0^2 \frac{\epsilon_0 w}{d}(1-K)$$

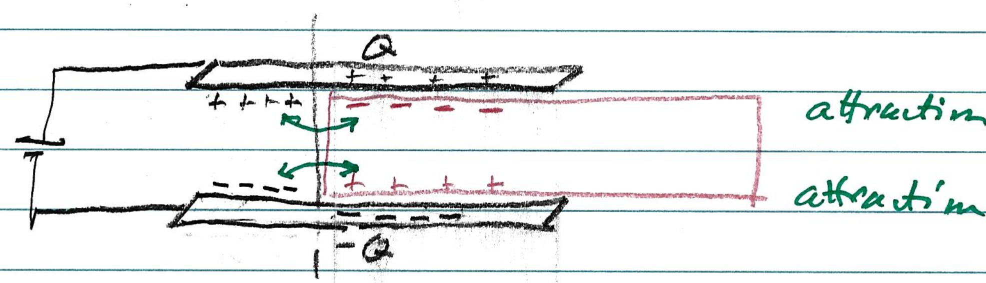
$F_x = -\frac{1}{2}V_0^2 \frac{\epsilon_0 w}{d}(K-1)$ in $-\hat{e}_x$ direction (2 pt)

(B) $U = \frac{1}{2} C V_0^2$

$$F_x = +\left(\frac{\partial U}{\partial x}\right)_V = \frac{1}{2}V_0^2 \frac{\partial C}{\partial x} = \frac{1}{2}V_0^2 \frac{\epsilon_0 w}{d}(1-K)$$

$F_x = -\frac{1}{2}V_0^2 \frac{\epsilon_0 w}{d}(K-1)$ same as (A). (2 pt)

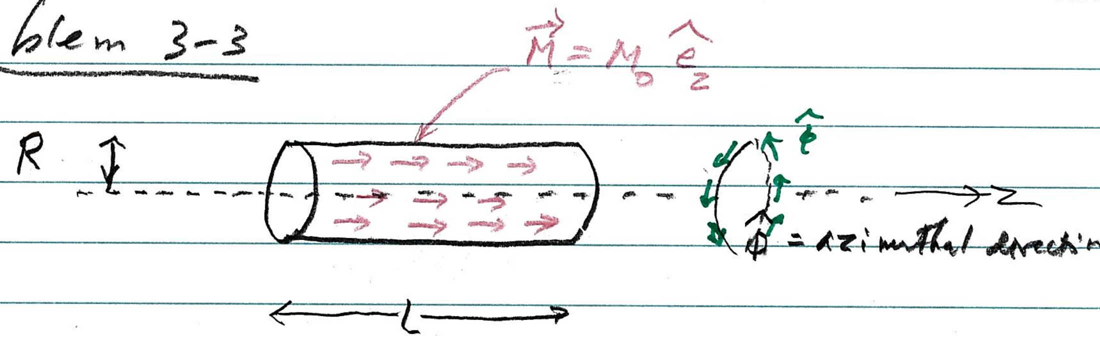
(C)



(2 pt)

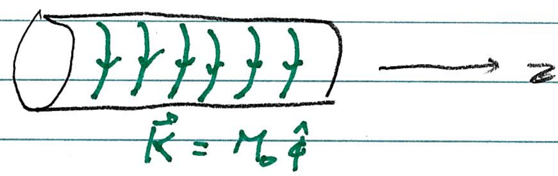
6 points

Problem 3-3



(A) $\vec{J}_M = \nabla \times \vec{M} = 0$

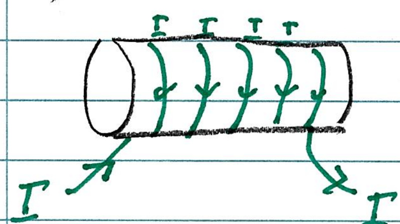
$$\vec{K}_M = \vec{M} \times \hat{n} = M_0\, \hat{e}_z \times \hat{r}_\perp = \boxed{M_0 \hat{\phi}}$$



(2 points)

(B) For a solenoid, $\vec{J} = 0$

$\vec{K} = nI\hat{\phi}$



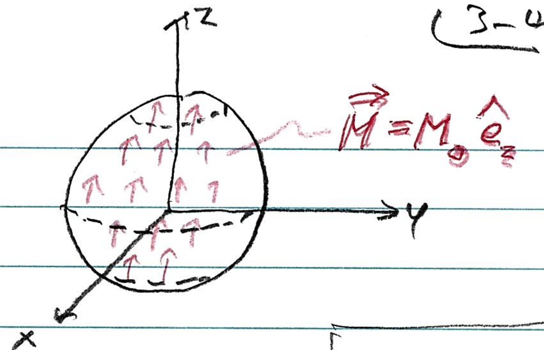
where n = density of windings

$$n = \frac{\delta N}{\delta l}$$

They are the same with $\boxed{M_0 = nI}$ (2 points)

4 points

Problem 3-4 (Terrella)



(A) $\vec{M} = M_0 \hat{e}_z$

$$\vec{J}_M = \nabla \times \vec{M} = 0$$

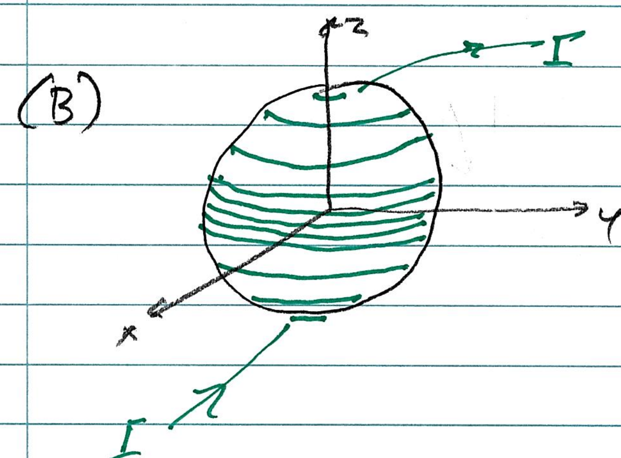
$$\vec{K}_M = \vec{M} \times \hat{n} = M_0 \hat{e}_z \times \hat{r} = \boxed{M_0 \sin\theta \, \hat{e}_\phi}$$

(2 points)

$$\begin{vmatrix} e_x & e_y & e_z \\ 0 & 0 & 1 \\ s\theta c\phi & s\theta s\phi & c\theta \end{vmatrix}$$

$\hat{e}_\phi = -\hat{e}_x \sin\phi + \hat{e}_y \cos\phi$

(B)

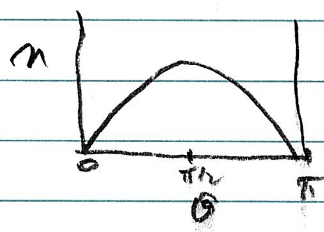


The wire carries current I.

The density of windings $= n = \frac{\delta N}{\delta \ell}$

$$\vec{K} = I n \hat{\phi}$$

So we need $n(\theta) = \dfrac{M_0 \sin\theta}{I}$ (2 points)



← denser windings at $\theta \approx \pi/2$.

4 points

Problem 3-5

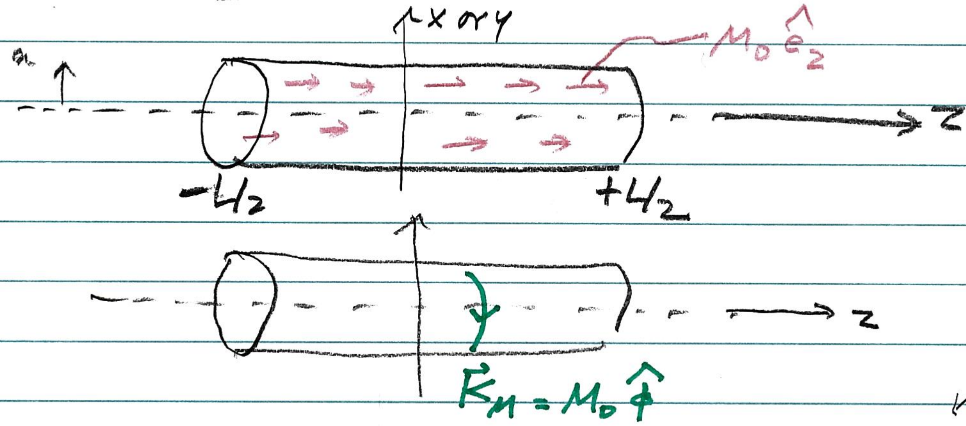
The "Terrella"

William Gilbert measured the magnetic field around a magnetized sphere, found that it had the same form as the magnetic field of the Earth, and published his famous result,

"Magnus magnes sive terristris globus"

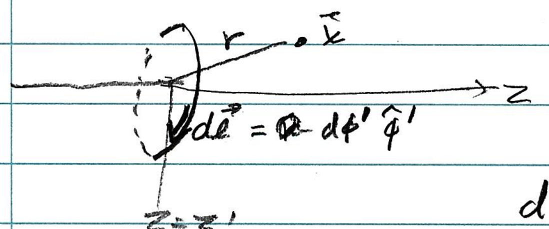
2 points

Problem 3-6 (Jackson Problem 5.19)



(x, y, z)

To determine the magnetic field at point $\vec{x}$ due to the current loop $\vec{K}_M dz'$ at $\vec{x}' = (a\cos\phi', b\sin\phi', z')$ (using cylindrical coordinates) Use the Biot-Savart law

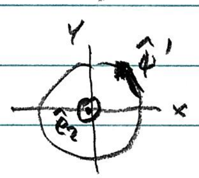


$$r = \sqrt{(\vec{r}_\perp - \vec{r}_\perp')^2 + (z-z')^2}$$

$$d\vec{B}(\vec{x}) = \frac{\mu_0 I}{4\pi r^3} \oint \left[d\vec{\ell} \times (\vec{x} - \vec{x}') \right]$$

On the z axis, $\vec{r}_\perp = 0$ and $\therefore r = \sqrt{r_\perp'^2 + (z-z')^2}$.

Also, $d\vec{\ell} \times (\vec{x} - \vec{x}') = a\, d\phi' \; \hat{\phi}' \times \left[z\hat{e}_z - \vec{r}_\perp' - z'\hat{e}_z \right]$



$$\hat{\phi}' \times \hat{e}_z = \hat{r}_\perp' \quad ; \quad \hat{\phi}' \times \vec{r}_\perp' = -\hat{e}_z\, r_\perp'$$

$$d\vec{B}(\vec{x}) = \frac{\mu_0 I \; a\, d\phi' \left\{ (z-z')\hat{r}_\perp' + \hat{e}_z r_\perp' \right\}}{4\pi \left[r_\perp'^2 + (z-z')^2 \right]^{3/2}} \qquad \boxed{r_\perp' = a}$$

Now integrate ϕ': 0 to 2π $\Rightarrow$

$$d\vec{B}(\vec{x}) = \frac{\mu_0 I}{4\pi} \left[a^2 + (z-z')^2 \right]^{-3/2} a \cdot 2\pi \cdot a\, \hat{e}_z$$

$$= \frac{\mu_0 I a^2}{2} \frac{\hat{e}_z}{\left[a^2 + (z-z')^2 \right]^{3/2}} \qquad \boxed{\begin{aligned} I &= K_\phi\, dz' \\ &= M_0\, dz' \end{aligned}}$$

Now integrate z' : $-L/2$ to $L/2$ $\Rightarrow$

$$\vec{B}(0,0,z) = \frac{\mu_0 M_0 a^2}{2}\,\hat{e}_z \int_{-L/2}^{L/2} \frac{dz'}{\left[a^2 + (z-z')^2\right]^{3/2}}$$

$$= \frac{\mu_0 M_0}{2}\,\hat{e}_z \left\{ \frac{L-2z}{\sqrt{4a^2 + (L-2z)^2}} + \frac{L+2z}{\sqrt{4a^2+(L+2z)^2}} \right\}$$

(2 pts)

Also,

$$\vec{H}(0,0,z) = \frac{1}{\mu_0}\vec{B} - \vec{M} \quad \text{where} \quad \vec{M} = M_0\,\hat{e}_z$$

$$= \frac{M_0}{2}\,\hat{e}_z \left\{ -2 + \frac{L-2z}{\sqrt{4a^2+(L-2z)^2}} + \frac{L+2z}{\sqrt{4a^2+(L+2z)^2}} \right\}$$ (2 pt)

$= \vec{H}$ inside the cylinder

(B) Plot $B_z(00z)/\mu_0 M_0$ and $H_z(00z)/M_0$ versus z,

using Mathematica, for $L/a = 5$.

(2 pt)

6 points

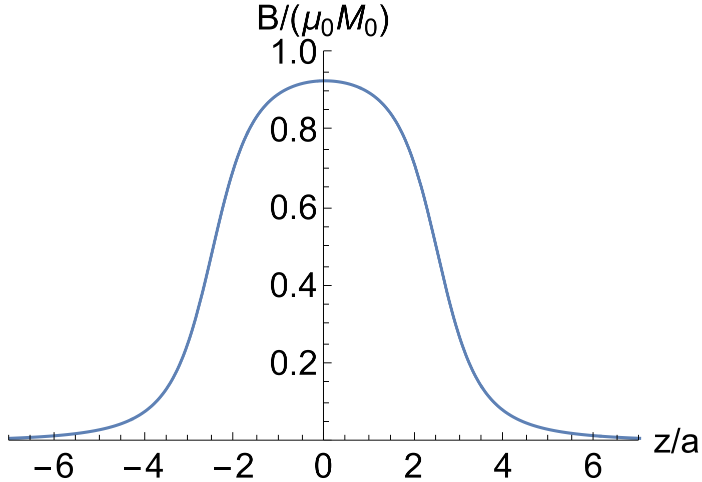
Problem 3-6 (B)

In[92]:=
```
Bz = 1 / 2 * ((L - 2 z) / Sqrt[4 a^2 + (L - 2 z)^2] +
      (L + 2 z) / Sqrt[4 a^2 + (L + 2 z)^2]);
f1[z_] = Bz /. {a → 1, L → 5}
Plot[f1[z] , {z, -7, 7}, PlotRange → {{-7, 7}, {0, 1}},
 AxesLabel → {"z/a", "B/(μ0M0)"}, BaseStyle → 18]
```

Out[93]=

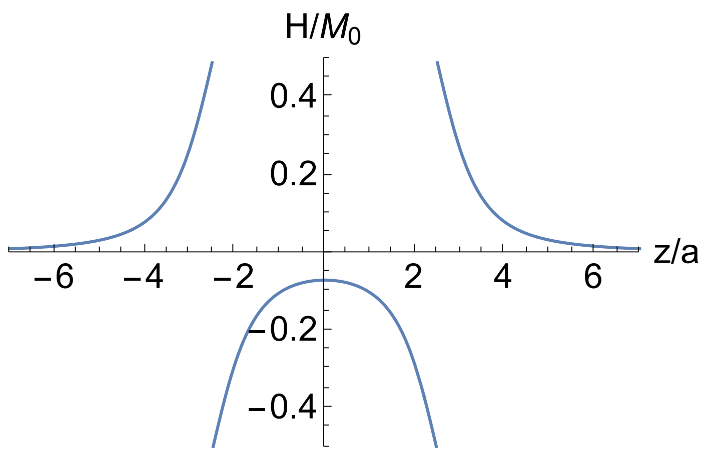
$$\frac{1}{2}\left(\frac{5-2z}{\sqrt{4+(5-2z)^2}}+\frac{5+2z}{\sqrt{4+(5+2z)^2}}\right)$$

Out[94]=




```
Plot[f1[z] +
   (-1) * HeavisideTheta[5 / 2 - z] * HeavisideTheta[5 / 2 + z],
  {z, -7, 7}, PlotRange → {{-7, 7}, {-0.5, 0.5}},
 AxesLabel → {"z/a", "H/M0"}, BaseStyle → 18]
```

Out[91]=



Problem 3-7

In[244]:=
```
(*A*) Integrate[Power[1 - m * Sin[ϕ] ^2, -1 / 2], ϕ];
Aa = (% /. ϕ → Pi / 2) - (% /. ϕ → 0)
(*B*) Integrate[Power[1 - m * Sin[ϕ] ^2, 1 / 2], ϕ];
Ba = (% /. ϕ → Pi / 2) - (% /. ϕ → 0)
(*C*) Integrate[Cos[ϕ] * Power[1 - m * Cos[ϕ], -1 / 2], ϕ];
Ca = (% /. ϕ → 2 Pi) - (% /. ϕ → 0) // Simplify
```

Out[245]= EllipticK[m]

Out[247]= EllipticE[m]

Out[249]=
$$\frac{4\left((-1+m)\,\mathrm{EllipticE}\left[\frac{2m}{-1+m}\right]+\mathrm{EllipticK}\left[\frac{2m}{-1+m}\right]\right)}{\sqrt{1-m}\;m}$$