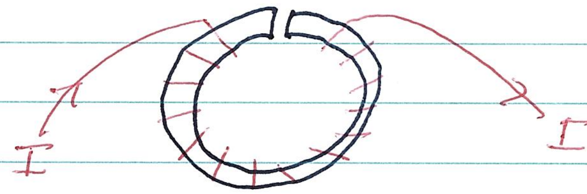
Homework Assignment #4

Problem 4-1

air gap ; d



toroidal electromagnet

radii = a, R

$\mu = 3000\,\mu_0$

$$\nabla \times \vec{H} = \vec{J} \quad \Rightarrow \quad \oint \vec{H}\cdot d\vec{s} = NI$$

$$\vec{H} \cong H_\phi \hat{\phi} \text{ where } (2\pi R - d) H_{IRON} + d H_{AIR} = NI$$

$$B_{IRON} \approx B_{AIR} \equiv B$$

$$H_{IRON} = \frac{B}{\mu} \text{ and } H_{AIR} = \frac{B}{\mu_0}$$

$$B\left[\frac{2\pi R - d}{\mu} + \frac{d}{\mu_0}\right] = NI$$

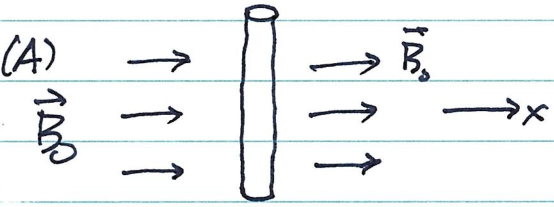
$$B = \frac{\mu_0 NI}{d + \frac{2\pi R - d}{3000}} \quad \text{in both air and iron}$$

$$H_{IRON} = \frac{NI}{3000\,d + 2\pi R - d} \quad \text{in the IRON}$$

$$H_{AIR} = \frac{NI}{d + \frac{2\pi R - d}{3000}} \quad \text{in the AIR GAP}$$

Problem 4-2

(A)



Boundary value problem with $\vec{J}_{free} = 0$; ignore the z-dependence ; polar coordinates $r_\perp$ and ϕ.

radius $= a$
length $= L$
permeability $= \mu$

$\nabla \times \vec{H} = 0 \Rightarrow \vec{H} = -\nabla \Phi_M$; and $\nabla^2 \Phi_M = 0$.

$$\Phi_M(r_\perp, \phi) = \begin{cases} C_0\, r_\perp \cos\phi & \text{for } r_\perp < a \\ \dfrac{-B_0}{\mu_0} r_\perp \cos\phi + \dfrac{C_1}{r_\perp}\cos\phi & \text{for } r_\perp > a \end{cases}$$

Boundary Conditions:

$\dfrac{\partial \Phi_M}{\partial \phi}$ is continuous at $r_\perp = a$ $\Rightarrow$ $C_0 = \dfrac{-B_0}{\mu_0} + \dfrac{C_1}{a^2}$

$\mu \dfrac{\partial \Phi_M}{\partial r_\perp}$ is continuous at $r_\perp = a$ $\Rightarrow$ $\mu C_0 = \mu_0 \left(\dfrac{-B_0}{\mu_0} - \dfrac{C_1}{a^2} \right)$

Solve for C_0 ; $C_0 = \dfrac{-2B_0/\mu_0}{\frac{\mu}{\mu_0} + 1} = \dfrac{-2B_0}{\mu + \mu_0}$

The magnetic induction inside the wire is

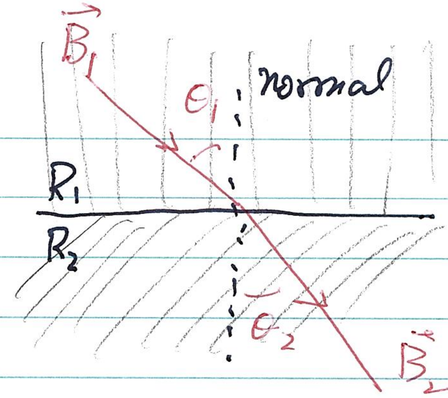
$$\vec{B}_{in} = -\mu \nabla \Phi_M = -\mu C_0 \hat{e}_x = \frac{2\mu}{\mu + \mu_0} B_0 \hat{e}_x$$

(B) $F_{IRON} = I L B$

$F = ILB$

$B_{IRON} \approx 2B_0$ and $B_{COPPER} \approx B_0 \Rightarrow \dfrac{F_{IRON}}{F_{COPPER}} \approx 2$.

Problem 4-3



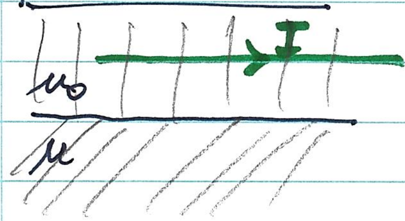
Boundary Conditions :

$$B_n^{(1)} = B_n^{(2)} \quad \Rightarrow \quad \mu_1 H_1 \cos\theta_1 = \mu_2 H_2 \cos\theta_2$$

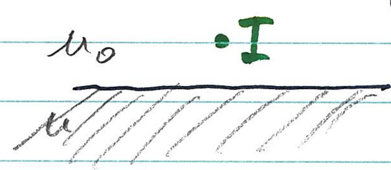
$$H_t^{(1)} = H_t^{(2)} \quad \Rightarrow \quad H_1 \sin\theta_1 = H_2 \sin\theta_2$$

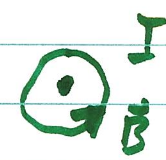
$$\therefore \quad \frac{1}{\mu_1} \tan\theta_1 = \frac{1}{\mu_2} \tan\theta_2 \quad \text{or} \quad \mu_1 \tan\theta_2 = \mu_2 \tan\theta_1$$

Problem 4-4

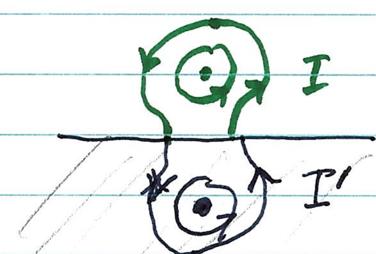


Use the result of Problem 4-3.





(A)



Assume $\mu = \infty$ $\Rightarrow$ $\tan\theta_1 = 0$
$(\mu_2 = \infty)$ $\quad\theta_1 = 0$

$\Rightarrow$ Image current I' parallel to I.

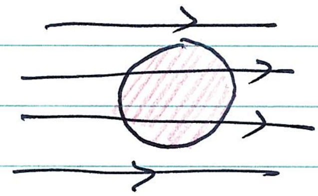
(B)

Assume $\mu = 0$ $\Rightarrow$ $\tan\theta_1 = \infty$
$(\mu_2 = 0)$ $\quad\theta_1 = 90$ degrees

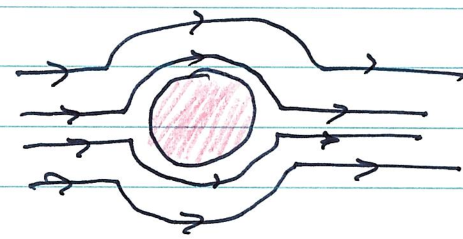
$\Rightarrow$ Image current I' antiparallel to I

Problem 4-5

The Meissner effect : a magnetic field is expelled from a superconductor



$T > T_c$



$T < T_c$

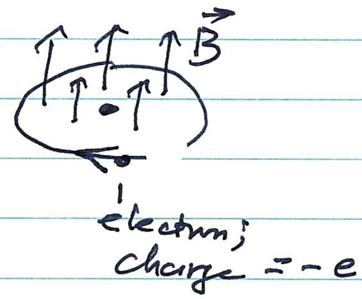
- As T is lowered to T_c the magnetic field is expelled from the superconducting sphere
- Superconductor surface currents create a magnetic field that cancels the applied field.
- Comparison to a magnetic material,

$$\vec{B} = \mu \vec{H} = 0 \quad \text{implies} \quad \mu = 0$$

(perfect diamagnet)

Problem 4-6

Diamagnetic susceptibility of neon at STP.



Langevin theory of diamagnetism

$$\Delta\vec{\mu} = -\frac{e^2}{6m}\langle r^2\rangle_{avg}\,\vec{B}$$

$$\vec{M} = N\langle\vec{m}\rangle = \chi_M \vec{H}$$

$$\therefore\ \chi_M = \frac{-e^2}{6m}\langle r^2\rangle_{avg}\, n_e N \mu_0$$

n_e = # of electrons
N = density of atoms

$$N = \frac{6.02\times10^{23}}{22.4\times10^{-3}\ m^3}\ ; \qquad n_e = 8 \text{ outer electrons}$$

$$\langle r^2\rangle_{avg} = R^2 \quad \text{where} \quad R = 0.4\times10^{-10}\ m$$

$$\chi_M = -2.03\times10^{-9}$$

(The experimental value is -3.69×10^{-9}.)

Problem 4.7 (Jackson problem 5-21)

(a) For permanent magnetization $\vec{M}$

$$\int_\infty \vec{B}\cdot\vec{H}\, d^3x = \int_\infty \underbrace{(\nabla\times\vec{A})\cdot\vec{H}}\, d^3x$$

$$\epsilon_{ijk}(\partial_j A_k) H_i$$
$$= \epsilon_{ijk}\left[\partial_j(A_k H_i) - A_k(\partial_j H_i)\right]$$
$$= \nabla\cdot(\vec{A}\times\vec{H}) + \vec{A}\cdot(\nabla\times\vec{H})$$

$$= \oint_{S\text{ at }\infty} \hat{n}\cdot(\vec{A}\times\vec{H})\, da + \int_\infty \vec{A}\cdot\vec{J}_{free}\, d^3x$$

$= 0$ (localized sources) $+ 0$ (no free current) $= 0$.

(B) The potential energy of a dipole in a magnetic field is $U = -\vec{m}\cdot\vec{B}$. For a distribution of dipoles,

$$W = -\sum_{\substack{i,j=1\\(i<j)}}^{N} \vec{m}_i\cdot\vec{B}_j \quad \text{(work to assemble N dipoles)}$$

$$= -\frac{1}{2}\sum_{\substack{i,j=1\\(i\neq j)}}^{N} \vec{m}_i\cdot\vec{B}_j = \frac{-1}{2}\sum_{ij=1}^{N} \vec{m}_i\cdot\vec{B}_j - \text{self energies}$$

IGNORE THE SELF ENERGY CONSTANT

For a continuum, $W = -\frac{1}{2}\int \vec{M}(\vec{x})\cdot\vec{B}(\vec{x})\, d^3x$

Now $\vec{B} = \mu_0(\vec{H}+\vec{M})$ and $\int_\infty \vec{B}\cdot\vec{H}\, d^3x = 0$

$$W = -\frac{1}{2}\mu_0\int \vec{M}\cdot\vec{H}\, d^3x - \frac{1}{2}\mu_0 \underbrace{\int \vec{M}(\vec{x})\cdot\vec{M}(\vec{x})\, d^3x}_{=\sum_i \int M_i^2 d^3x}$$

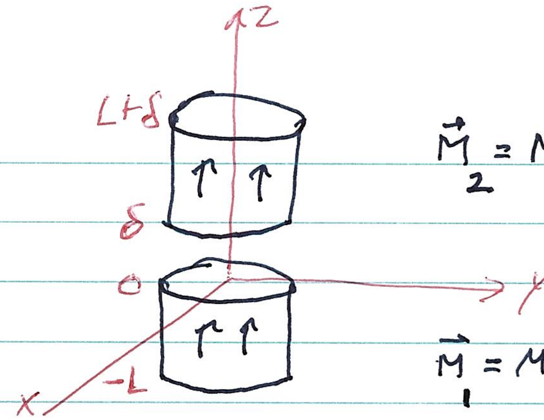
ANOTHER SELF ENERGY CONSTANT

$$W = -\frac{1}{2}\mu_0\int \vec{M}\cdot\vec{H}\, d^3x + \text{constant}$$

Problem 4-8

Forces on 2 permanent magnets

(NOT linear media)



$\vec{M}_2 = M_0 \hat{e}_z$

$\vec{M}_1 = M_0 \hat{e}_z$

Calculate the energy $U(\delta)$. Then the force on the upper magnet is $\vec{F} = F_z \hat{e}_z$ where $F_z = -\frac{\partial U}{\partial \delta}$.

From Problem 4-7 we have

$$U_{total} = -\frac{1}{2}\mu_0 \int (\vec{M}_1 + \vec{M}_2)\cdot(\vec{H}_1 + \vec{H}_2)\, d^3x$$

$$= -\frac{1}{2}\mu_0 \int \left[\vec{M}_1\cdot\vec{H}_1 + \vec{M}_1\cdot\vec{H}_2 + \vec{M}_2\cdot\vec{H}_1 + \vec{M}_2\cdot\vec{H}_2 \right] d^3x$$

equal

these are self energies, independent of δ; IGNORE THESE CONSTANTS

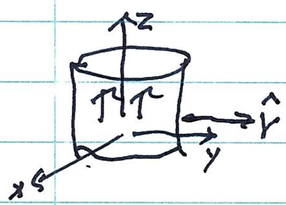
$$U = -\mu_0 \int \vec{M}_2\cdot\vec{H}_1\, d^3x$$

Now, $\vec{H}_1 = \frac{1}{\mu_0}\vec{B}_1 - \vec{M}_1$ and $\int \vec{M}_2\cdot\vec{M}_1\, d^3x = 0$ because the magnets do not overlap; also, $\vec{B}_1 = \nabla\times\vec{A}_1$.

Therefore
$$U = -\int \underbrace{\vec{M}_2\cdot(\nabla\times\vec{A}_1)}_{=\nabla\cdot(-\vec{M}_2\times\vec{A}_1) + \vec{A}_1\cdot\underbrace{(\nabla\times\vec{M}_2)}_{=0}}\, d^3x$$

$$U = \oint_{S_2} \hat{n}\cdot(\vec{M}_2\times\vec{A}_1)\, da_2 \quad \text{by Gauss's theorem}$$

Or, $\quad U = \oint_{S_2} (\hat{n}\times\vec{M}_2)\cdot\vec{A}_1\, da_2$



$$= \oint_{S_2} -M_0\, \hat{e}_\phi(\phi_2)\cdot\vec{A}_1\, da_2$$

Use cylindrical coordinates, $\{r_\perp, \phi, z\}$

Vector Potential To determine $\vec{A}_1(r_\perp, \phi, z)$, treat the lower magnet as a stack of circular current loops; bound current density $\vec{K}_1 = M_0\, \hat{e}_\phi(\phi_1)\, \delta(r_{1\perp}-a)$

$$\vec{A}_1(r_\perp, \phi, z) = \frac{\mu_0}{4\pi}\int \frac{\hat{e}_\phi(\phi_1)\, M_0\, \delta(r_{1\perp}-a)\; r_{1\perp} dr_{1\perp}\, d\phi_1\, dz_1}{|\vec{x}-\vec{x}_1|}$$

$$= \frac{\mu_0 M_0\, a}{4\pi}\int \frac{\hat{e}_\phi(\phi_1)\, d\phi_1\, dz_1}{|\vec{x}-\vec{x}_1|} \qquad \begin{array}{l}\phi_1 : 0 \to 2\pi \\ z : -L \to 0\end{array}$$

By symmetry, $\vec{A}_1(r_\perp, \phi, z) = A_{1\phi}(r_\perp, z)\, \hat{e}_\phi(\phi)$

$$A_{1\phi}(r_\perp, z) = \frac{\mu_0 M_0\, a}{4\pi}\int \frac{\cos(\phi-\phi_1)\, d\phi_1\, dz_1}{|\vec{x}-\vec{x}_1|}$$

$$|\vec{x}-\vec{x}_1| = \sqrt{\vec{x}^2 + \vec{x}_1^2 - 2\vec{x}_1\cdot\vec{x}}$$

$$= \left(r_\perp^2 + z^2 + r_{1\perp}^2 + z_1^2 - 2 r_\perp r_{1\perp}\cos(\phi-\phi_1) - 2 z z_1 \right)^{1/2}$$

$$= \left((z-z_1)^2 + r_\perp^2 + a^2 - 2 r_\perp a \cos(\phi-\phi_1) \right)^{1/2}$$

WLOG set $\phi = 0$.

We need $\vec{A}_1(r_1, \phi, z)$ on S_2 where $r_1 = a$, WLOG $\phi = 0$, $z = z_2$. Also, $da_2 = r_{21}\, d\phi_2\, dz_2 = a\, d\phi_2\, dz_2$.

Also, $\int d\phi_2 \to 2\pi$. Thus

$$U = -M_0 \quad 2\pi a \int_\delta^{L+\delta} dz_2\ A_{1\phi}(a, z_2)$$

$$= -M_0 \cdot 2\pi a \ \frac{\mu_0 M_0 a}{4\pi} \int_\delta^{L+\delta} dz_2 \int_0^{2\pi} d\phi_1 \int_{-L}^{0} dz_1 \ \cos(\phi_1)$$

$$\left[(z_2 - z_1)^2 + 2a^2 - 2a^2 \cos(\phi_1) \right]^{-1/2}$$

$$U = -\frac{1}{2}\mu_0 M_0 a^2 \int_\delta^{L+\delta} dz_2 \int_0^{2\pi} d\phi_1 \int_{-L}^{0} dz_1 \ \frac{\cos\phi_1}{\sqrt{(z_1 - z_2)^2 + 2a^2(1 - \cos\phi_1)}}$$

The force on the upper magnet (M_2) is $F_z \hat{e}_z$

$$F_z = \frac{-\partial U}{\partial \delta} = \frac{1}{2}\mu_0 M_0 a^2 \int_0^{2\pi} d\phi_1 \int_{-L}^{0} dz_1 \ \cos\phi_1$$

$$\left\{ \left[(z_1 - L - \delta)^2 + 2a^2(1 - \cos\phi_1) \right]^{-1/2} \right.$$

$$\left. - \left[(z_1 - \delta)^2 + 2a^2(1 - \cos\phi_1) \right]^{-1/2} \right\}$$

$z_1 - L = -\xi$

$z_1 = -\xi$

$$F_z = \frac{1}{2}\mu_0 M_0 a^2 \int_0^{2\pi} d\phi_1 \cos\phi_1$$

$$\left\{ \int_L^{2L} d\xi \left[(\xi + \delta)^2 + 2a^2(1 - \cos\phi_1) \right]^{-1/2} \right.$$

$$\left. - \int_0^L d\xi \left[(\xi + \delta)^2 + 2a^2(1 - \cos\phi_1) \right]^{-1/2} \right\}$$

Evaluation of Integrals

$$F_z = -\tfrac{1}{2}\mu_0 M_0 a^2 \int_0^{2\pi} d\phi_1 \cos\phi_1$$

$$\left\{ \int_0^L d\zeta \left[(\zeta+\delta)^2 + 2a^2(1-\cos\phi_1) \right]^{-1/2} \right.$$

$$\left. - \int_L^{2L} d\zeta \left[(\zeta+\delta)^2 + 2a^2(1-\cos\phi_1) \right]^{-1/2} \right\}$$

The integrals can be evaluated using Mathematica. Do the ζ integrals first.

$$F_z = -\tfrac{1}{2}\mu_0 M_0 a^2 \int_0^{2\pi} d\phi_1 \cos\phi_1$$

$$\left\{ \mathrm{ArcTanh}\left(\frac{L+\delta}{\sqrt{q^2+(L+\delta)^2}}\right) - \mathrm{ArcTanh}\left(\frac{\delta}{\sqrt{q^2+\delta^2}}\right) \right.$$

$$\left. - \mathrm{ArcTanh}\left(\frac{2L+\delta}{\sqrt{q^2+(2L+\delta)^2}}\right) + \mathrm{ArcTanh}\left(\frac{L+\delta}{\sqrt{q^2+(L+\delta)^2}}\right) \right.$$

where $q^2 = 2a^2(1-\cos\phi_1)$.

Now do the ϕ_1 integrals.

$$F_z = -\tfrac{1}{2}\mu_0 M_0 a^2 \left\{ 2G(L+\delta) - G(\delta) - G(2L+\delta) \right\}$$

$$G(\Delta) = \tfrac{1}{2}\delta^2 \left[K\left(\frac{-4a^2}{\delta^2}\right) - E\left(\frac{-4a^2}{\delta^2}\right) \right] + 2a^2 K\left[-\frac{4a^2}{\delta^2}\right]$$

elliptic integral functions

Analytic calculations for Problem 4 - 8

Using the hints that I handed out in class Wednesday Sept 25,
the interaction potential energy of the two magnets is

$$U = -\frac{1}{2}\,\mu_0\, M_0\, a^2 \int_\delta^{L+\delta} \mathrm{dz}_2 \int_{-L}^{0} \mathrm{dz}_1 \int_0^{2\pi} \mathrm{d}\phi_1 \,\frac{\cos\phi_1}{[\,(z_1-z_2)^2+2\,a^2(1-\cos\phi_1)]^{1/2}}$$

The force on the upper magnet (M_2) is $F_z\,\hat{e}_z$ where

$$F_z = -\frac{\partial U}{\partial \delta}$$

$$= \frac{1}{2}\,\mu_0\, M_0\, a^2 \mathrm{dz}_1 \int_0^{2\pi} \mathrm{d}\phi_1 \cos\phi_1 \int_{-L}^{0} \mathrm{dz}_1$$

$$\{\,[\,(z_1 - \mathrm{L} - \delta)^2 + 2\,a^2(1-\cos\phi_1)\,]^{-1/2}$$

$$-\,[\,\,(z_1-\delta)^2 + 2\,a^2(1-\cos\phi_1)\,]^{-1/2}\,\}$$

Evaluation of the z integrals

The result is

$$F_z = -\frac{1}{2}\,\mu_0\, M_0\, a^2 \int_0^{2\pi} \mathrm{d}\phi_1 \cos\phi_1 \,\{\,2\,\mathrm{ArcTanh}\left(\frac{L+\delta}{\sqrt{q^2+(L+\delta)^2}}\right) - \mathrm{ArcTanh}\left(\frac{\delta}{\sqrt{q^2+\delta^2}}\right)$$

$$-\,\,\mathrm{ArcTanh}\left(\frac{2\,\mathrm{L}+\delta}{\sqrt{q^2+(2\,\mathrm{L}+\delta)^2}}\right)\}$$

where $q^2 = 2\,a^2\,(1-\cos\phi_1)$.

Now do the ϕ_1 integrations.

The force on the upper magnet (M_2) is

$$\mathrm{F_z}(\delta) = -\,1/2\,\mu_0\,\mathrm{M}_0\,\mathrm{a}^2\,\int_0^{2\pi} \mathrm{d}\phi_1 \cos\phi_1 \{\,2^*\;\mathrm{Z(L}+\delta) - \mathrm{Z}(\delta) - \mathrm{Z(2L}+\delta)\,\}$$

where $\mathrm{Z}(\lambda) = \mathrm{ArcTanh}\left[\frac{\lambda}{\sqrt{\mathrm{q}^2+\lambda^2}}\right]$ and $\mathrm{q}^2 = 2\,\mathrm{a}^2\,(1-\cos(\phi_1))$.

From now on set $\mu_0\,\mathrm{M}_0 = 1$.

In[1]:=
```
F[ϕ1_] = ArcTanh[λ / Sqrt[λ^2 + 2 a^2 * (1 - Cos[ϕ1])]];
temp = Integrate[Cos[ϕ1] * F[ϕ1], ϕ1];
basicI = -(a^2/2) * ((temp /. {ϕ1 → 2 Pi}) - (temp /. {ϕ1 → 0}));
basicI = basicI // Expand // PowerExpand // FullSimplify
```

Out[4]=
$$\frac{1}{2}\left(\lambda^2 \text{EllipticE}\left[-\frac{4 a^2}{\lambda^2}\right] - (4 a^2 + \lambda^2) \text{EllipticK}\left[-\frac{4 a^2}{\lambda^2}\right]\right)$$

In[5]:=
```
G[Λ_] = basicI /. {λ → Λ}
force[δ_] = 2 * G[L + δ] - G[δ] - G[2 L + δ];
nforce[δ_] = force[δ] /. {a → 1, L → 2};
Plot[nforce[δ], {δ, 0, 4},
  Frame → True, FrameLabel → {"δ", "Fz(δ) ;  a=1, L=2"},
  BaseStyle → {18}, AspectRatio → 1, ImageSize → Medium] // Rasterize
```

Out[5]=
$$\frac{1}{2}\left(\Lambda^2 \text{EllipticE}\left[-\frac{4 a^2}{\Lambda^2}\right] - (4 a^2 + \Lambda^2) \text{EllipticK}\left[-\frac{4 a^2}{\Lambda^2}\right]\right)$$

Out[8]=