Problem 6-1

PLANE WAVE W/ $I = 1360\ W/m^2$

$$I = \langle S_z \rangle = \frac{E_0 B_0}{2\mu_0} \quad \text{and} \quad B_0 = E_0/c$$

(A) $E_0 = \sqrt{2\mu_0 c I}$ and $B_0 = E_0/c$ (2)

(B) $I = 1360\ W/m^2 \Rightarrow E_0 = 1013\ V/m$ and $B_0 = 3.38 \times 10^{-6}\ T$ (2)

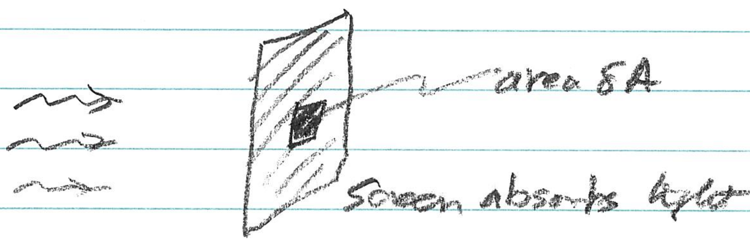
(C) $$\frac{\delta N}{\delta A \delta t} = \frac{\delta U/\hbar\omega}{\delta A \delta t} = \frac{I}{\hbar\omega} \quad \text{and} \quad \omega = 2\pi \frac{c}{\lambda}$$

$$\frac{\delta N}{\delta A \delta t} = \frac{I\lambda}{2\pi \hbar c} = 3.42 \times 10^{21}\ m^{-2} s^{-1}$$ (2)

6 points

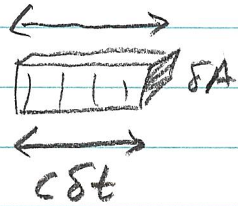
Problem 6-2 (Jackson 6-8) RADIATION PRESSURE

(A)



Conservation of momentum: $\delta P = F\,\delta t$

$$p_{\text{ressure}} = p = \frac{F}{\delta A} = \frac{\delta P}{\delta A \cdot \delta t}$$



$$\delta P = g\,\delta V = g\,\delta A\, c\,\delta t$$

$$\therefore \text{pressure} = gc$$

where the momentum density $g = \frac{S}{c^2}$; $p = \frac{S}{c}$

Energy flux $S\,\delta A\,\delta t = u\,\delta A \cdot c\,\delta t \Rightarrow S = uc$

Thus $p = u$. 2 points

(B) $S = 1400\ \text{W/m}^2 \Rightarrow p = 4.67 \times 10^{-6}\ \frac{\text{N}}{\text{m}^2}$

Mass $= \sigma_{mass}\,\delta A \Rightarrow$ acceleration $= \frac{F}{\text{mass}} = \frac{p\,\delta A}{\sigma_{mass}\,\delta A} = \frac{p}{\sigma_{mass}}$

acceleration $= 4.67 \times 10^{-3}\ \text{m/s}^2$ 2 points

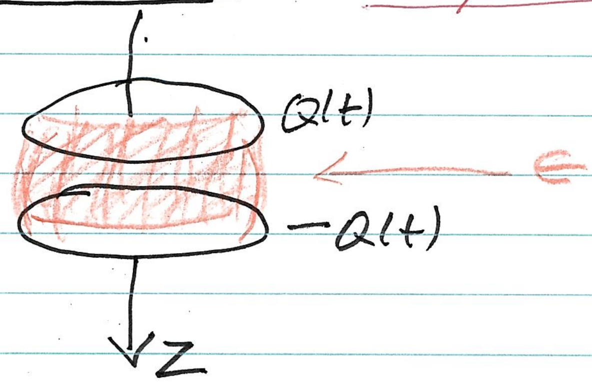
According to Wikipedia, $P_{\text{solar wind}} \approx (1-6) \times 10^{-9}\ \text{N/m}^2$

about $10^{-3}\ P_{\text{radiation}}$ 1 point

5 points

Problem 6-5

LEAKY CAPACITOR



(A) Quasi static solution $E_z = \dfrac{Q}{\epsilon \pi a^2}$

$$J_z = g E_z = \frac{g Q}{\epsilon \pi a^2}$$

$$\frac{dQ}{dt} = -J_z \pi a^2 = -\frac{gQ}{\epsilon}$$

$$\therefore Q(t) = Q_0 e^{-gt/\epsilon}$$

2 points

(B) Displacement current density

$$\vec{J}_D = \epsilon \frac{\partial \vec{E}}{\partial t} = \frac{1}{\pi a^2} \dot{Q} \hat{e}_z = -\frac{g}{\epsilon} \frac{Q}{\pi a^2} \hat{e}_z$$

2 points

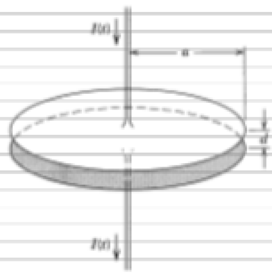
(C) $\vec{J} + \vec{J}_D = 0$ so the magnetic field is ZERO.

2 points

6 points total

Jackson Problem 6.14

6.14 An ideal circular parallel plate capacitor of radius a and plate separation $d \ll a$ is connected to a current source by axial leads, as shown in the sketch. The current in the wire is $I(t) = I_0 \cos \omega t$.



Problem 6.14

(a) Calculate the electric and magnetic fields between the plates to second order in powers of the frequency (or wave number), neglecting the effects of fringing fields.

(b) Calculate the volume integrals of w_e and w_m that enter the definition of the reactance X, (6.140), to second order in ω. Show that in terms of the input current I_i, defined by $I_i = -i\omega Q$, where Q is the *total charge* on one plate, these energies are

$$\int w_e \, d^3x = \frac{1}{4\pi\epsilon_0} \frac{|I_i|^2 d}{\omega^2 a^2}, \qquad \int w_m \, d^3x = \frac{\mu_0}{4\pi} \frac{|I_i|^2 d}{8} \left(1 + \frac{\omega^2 a^2}{12c^2}\right)$$

(c) Show that the equivalent series circuit has $C = \pi\epsilon_0 a^2/d$, $L = \mu_0 d/8\pi$, and that an estimate for the resonant frequency of the system is $\omega_{res} \approx 2\sqrt{2}\, c/a$. Compare with the first root of $J_0(x)$.

The problems depends on Section 6.9, so start by reading that. In particular, we will need to use complex fields and sources.

The current is I(t) = $I_0 \cos(\omega t)$ = Re $I_o\, e^{-i\omega t}$.

Do the calculations for $I_i(t) = I_0\, e^{-i\omega t}$, with *the real part being understood*. Also, Q(t) = $\frac{I_0}{\omega}$ $\sin(\omega t)$ = Re i $\frac{I_0}{\omega}\, e^{-i\omega t}$.

So the charge in the complex formulation is q = i $I_0/\omega\; e^{-i\omega t}$.

Part (A)

Sources:

$Q(t) = Q_0\, e^{-i\omega t}$ and $I(t) = I_0\, e^{-i\omega t}$ (Re understood)

where $I = \dot{Q} = -i\omega\, Q_0\, e^{-i\omega t}$; $\Longrightarrow Q_0 = i\, I_0 / \omega$.

Fields:

(i) Electrostatics approximation, $C = \frac{\epsilon_0 \pi a^2}{d}$

$E_z = \frac{\Delta V}{d} = \frac{Q}{d\,C} = i\, \frac{I}{\omega \pi a^2 \epsilon_0}$

(ii) Displacement current $J_z = \epsilon_0 \frac{\partial E_z}{\partial t} = \frac{I}{\pi a^2}$

$\nabla\times \vec{H} = \vec{J}$ implies $H_\phi = \frac{\pi r^2}{2 \pi r} J_z = \frac{r}{2} \frac{I}{\pi a^2}$

(iii) add the electric field from Faraday's law,

$\nabla\times \vec{E} = i\, \omega\, \vec{B} = i\omega\mu_0\, \vec{H} = i\omega\mu_0\, \frac{I}{\pi a^2} \frac{r}{2}\, \hat{e}_\phi$

$E_z = \frac{i\, \omega \mu_0\, I}{\pi\, a^2} \frac{r^2}{4}$ (sign in uncertain)

combining,

$E_z = \frac{i\, I}{\omega\, \pi\, a^2\, \epsilon_0} \{ 1 - \frac{\omega^2\, r^2}{4\, c^2} + \text{higher order} \}$

negative sign for Lenz's law

(iv) Recalculate H from the new displacement current

$H_\phi = \frac{I}{\pi\, a^2} \frac{1}{2\, \pi\, r} \int_0^r (1 - \frac{\omega^2\, r'^2}{4\, c^2})\, 2\pi\, r' dr'$

$H_\phi = \frac{I}{\pi\, a^2} (\frac{r}{2} - \frac{\omega^2}{4\, c^2} \frac{r^3}{4}) = \frac{I}{\pi\, a^2} \frac{r}{2} \{ 1 - \frac{\omega^2\, r^2}{8\, c^2} + \text{h.o.} \}$

Part (B)

We have $w_e = \frac{1}{4}\vec{E}\cdot\vec{D}^*$ and $w_m = \frac{1}{4}\vec{B}\cdot\vec{H}^*$.

$w_e = \frac{\epsilon_0}{4}\,|E|^2 = |I|^2 \frac{\epsilon_0}{4(\epsilon_0\,\omega\,\pi\,a^2)^2}\{1 - \frac{\omega^2 r^2}{2c^2} + \text{ho}\}$

$w_m = \frac{\mu_0}{4}\,|H|^2 = |I|^2 \frac{\mu_0}{4(\pi\,a^2)^2}\frac{r^2}{4}\{1 - \frac{\omega^2 r^2}{4c^2} + \text{ho}\}$

Now integrate over the volume;

$\int d^3x = \int 2\pi\,r\,dr\,dz = 2\pi\,d\int r\,dr;$

$U_e = \int w_e\,d^3x = |I/\omega|^2 \frac{\epsilon_0}{4(\epsilon_0\,\pi\,a^2)^2} 2\pi d \frac{a^2}{2}\{1 - \frac{\omega^2 a^2}{4c^2}\}$

$= \frac{q^2 d}{4\pi\epsilon_0 a^2}\{1 - \frac{\omega^2 a^2}{4c^2}\}$ where $q = I/\omega$

$U_m = \int w_m\,d^3x = |I/\omega|^2 \frac{\mu_0\,\omega^2}{4(\pi\,a^2)^2} 2\pi d \frac{a^4}{16}\{1 - \frac{\omega^2 a^6}{6c^2}\}$

$= \frac{\mu_0\,\omega^2 q^2 d}{32\pi}\{1 - \frac{\omega^2 a^2}{6c^2}\}$

Finally, Q = total charge on either plate.

Calculate Q from Gauss's theorem,

$\int_0^a E_z\,2\pi\,r\,dr = \frac{Q}{\epsilon_0}$.

Therefore $Q = \int_0^a \frac{i\,I}{\omega\,\pi a^2}(1 - \frac{\omega^2 r^2}{4c^2})\,2\pi\,r\,dr$

$= \frac{i\,I}{\omega\,\pi a^2}2\pi(\frac{a^2}{2} - \frac{\omega^2 a^4}{16c^2}) = i\,q\,(1 - \frac{\omega^2 a^2}{8c^2})$ [recall $q=I/\omega$]

$|Q|^2 = |q|^2(1 - \frac{\omega^2 a^2}{4c^2})$

$|q|^2 = |Q|^2(1 + \frac{\omega^2 a^2}{4c^2})$ neglecting $O(\omega^4)$

$U_e = \frac{q^2 d}{4\pi\epsilon_0 a^2}\{1 - \frac{\omega^2 a^2}{4c^2}\} = \frac{Q^2 d}{4\pi\epsilon_0 a^2}$ neglecting $O(\omega^4)$

$U_m = \frac{\mu_0\,\omega^2 q^2 d}{32\pi}\{1 - \frac{\omega^2 a^2}{6c^2}\} = \frac{\mu_0\,\omega^2 Q^2 d}{32\pi}\{1 + \frac{\omega^2 a^2}{12c^2}\}$

neglecting $O(\omega^4)$

Part (C)

Calculate the reactance

$X = \frac{4\,\omega}{|I|^2}\,(U_m - U_e) = \frac{4\,\omega}{Q^2\,\omega^2}\,(U_m - U_e)$

$X \approx \omega\,\frac{\mu_0\,d}{8\,\pi} - \frac{1}{\omega}\,\frac{d}{\pi\,\epsilon_0\,a^2} = \omega\,L - \frac{1}{\omega\,C}$

I.e., it is approximately equivalent to an LC circuit, with inductance $L = \frac{\mu_0\,d}{8\,\pi}$ and capacitance $C = \frac{\epsilon_0\,\pi\,a^2}{d}$

The resonant frequency is $\frac{1}{\sqrt{LC}} \approx \frac{2\sqrt{2}\,c}{a}$.

Comparison:

$2\sqrt{2} = 2.82$ compare to first root of $J_0(x) = 2.40$.