Homework Assignment 8 due Monday October 28

To aid in grading, draw a box around your answer, using a red pencil.

8-1. Use Fresnel's equations and Snell's law.
Show that at Brewster's angle the reflected and transmitted waves are perpendicular.

Brewster's angle: When $\theta = \theta_B$, there is no reflection for TM polarization.

The TM Fresnel coefficient is N/D where

$$N = (n')^2\cos(\theta) - n\sqrt{(n')^2 - n^2\sin^2(\theta)} = (n')^2\cos(\theta) - n\,n'\cos(\theta')$$

$$D = (n')^2\cos(\theta) + n\sqrt{(n')^2 - n^2\sin^2(\theta)}$$

Reflection = 0 $\Longrightarrow$ $n'\cos(\theta_B) = n\cos(\theta')$

Also, Snell's law is $n\sin(\theta_B) = n'\sin(\theta')$

Combine these equations $\Longrightarrow \frac{\cos(\theta')}{\cos(\theta)} = \frac{\sin(\theta)}{\sin(\theta')}$;

$\therefore\ \theta' = \frac{\pi}{2} - \theta$; or, $\theta + \theta' = 90$ degrees.

8-2. A fish in a lake sees the entire sky compressed into a cone.
Calculate the cone angle ≡ the angle from the center axis to the cone surface,
in degrees.

The cone angle is equal to the critical angle for total internal reflection of light
on a water→air interface.

∴ θ_{cone} = arcsin[n_{air} / n_{water}] = arcsin[1 / 1.33] = 48.8 degrees.

8-3. (a) Derive the wave equation for an electromagnetic wave in a material with permittivity ϵ and conductivity g. (Ignore magnetization.) (b) Solve the equation for a plane wave with frequency ω. (c) Calculate the absorption length.

(A) $\nabla\times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$ and $\nabla\times \vec{H} = g\,\vec{E} + \epsilon\,\frac{\partial \vec{E}}{\partial t}$.

Evaluate $\nabla\times(\nabla\times\vec{E})$ and simplify $\Rightarrow \mu_0\,\epsilon\,\frac{\partial^2 \vec{E}}{\partial t^2} - \nabla^2\vec{E} + \mu_0\,g\,\frac{\partial \vec{E}}{\partial t} = 0$

(B) A plane wave solution requires $k^2 = \mu_0\epsilon\omega^2 + i\,\mu_0 g\omega$.

(C) Write $k = \beta + \frac{i\alpha}{2}$ where α is the absorption coefficient.

Absorption length = $\delta = 1/\alpha$.

Algebra $\Rightarrow \alpha = \sqrt{2\,\epsilon\,\mu_0\,\omega^2}\;[\,-1 + \sqrt{1 + (g/\epsilon\omega)^2}\,]^{1/2}$

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In[∘]:= (*algebra by Mathematica*)
sol = Solve[{β^2 - α^2 / 4 == μ0 * ϵ * ω^2, α * β == μ0 * g * ω}, {α, β}];
Length[sol]
AC = α /. sol[[4]];
AC = AC /. μ0 → ρ / (2 * ϵ * ω^2) // FullSimplify;
AC = (AC /. ρ → 1) * ρ
```

Out[∘]= 4

Out[∘]= $\rho\sqrt{-1 + \sqrt{1 + \frac{g^2}{\epsilon^2\,\omega^2}}}$

8-4. Jackson Problem 7.5
This problem answers the question "Why are metals shiny".

Solution by Mathematica is attached.

8-5. Jackson Problem 7.13.

Solution by Mathematica is attached.

8-6. The index of refraction of diamond is 2.417 for $\lambda = 589.3$ nm.
The static dielectric constant is 5.50. Consider the Lorentz model
for dispersion, assuming a single resonant frequency with $\gamma/\omega_0 \longrightarrow 0$.
Calculate $\hbar\omega_0$ in eV.

Given two pieces of data, we can determine two model parameters.

The Lorentz model for dispersion, with $\gamma = 0$ ...

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{N e^2}{\epsilon_0 m} f_0 \left[\omega_0^2 - \omega^2 \right]^{-1}$$

$$n(\omega) = \sqrt{\frac{\epsilon(\omega)}{\epsilon_0}}$$

where f_0 and ω_0 are coefficients to be determined from the data.

Parameters ...

```
In[∘]:= Remove["Global`*"]
rho = 3.51*^3 * kg / meter ^ 3;
MC = 12.01 * amu /. {amu → 1.66*^-27 * kg};
NC = rho / MC;
e = 1.6*^-19 * coulomb;
m = 9.1*^-31 * kg;
cl = 3.0*^8 * meter / second;
ϵ0 = 1 / (μ0 * c ^ 2) /. {μ0 → 4 * Pi * 1.0*^-7, c → 3.0*^8};
ϵ0 = ϵ0 * coulomb ^ 2 / meter / joule;
const = NC * e ^ 2 / (ϵ0 * m) /. {joule → kg * meter ^ 2 / second ^ 2}
```

Out[∘]= $\frac{5.60153 \times 10^{32}}{\text{second}^2}$

First equation: ϵ / ϵ_0 = dc = 5.50 at $\omega = 0$

```
In[∘]:= dc = 5.50;
sol = Solve[1 + const * f0 / ω0 ^ 2 == dc, f0];
f0v = f0 /. sol[[1]]
```

Out[∘]= 8.03351×10^{-33} second^2 $\omega0^2$

Second equation: $\sqrt{\epsilon/\epsilon_0}$ = nref = 2.417 at $\omega = 2\pi c/(589.3\ \mathrm{nm})$

In[∘]:=
```
nref = 2.417;
ω1 = 2 * Pi * cl / (589.3*^-9 * meter);
EQ = {1 + const * f0v / (ω0^2 - ω1^2) == nref^2};
sol2 = Solve[EQ, ω0];
ω0v = ω0 /. sol2[[2]];
hbar = 197.0*^-9 * eV * meter / cl;
resonantenergy = hbar * ω0v
```

Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

Out[∘]= 7.9045 eV

The result is $\hbar\omega_0$ = 7.9045 eV .

Go back and evaluate f_0!

In[∘]:=
```
f0v
ω0v
f0v /. { ω0 → ω0v}
```

Out[∘]= $8.03351 \times 10^{-33}\ \mathrm{second}^2\ \omega 0^2$

Out[∘]= $\frac{1.20373 \times 10^{16}}{\mathrm{second}}$

Out[∘]= 1.16403

which makes sense.