Name SOLUTIONS

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INSTRUCTIONS:

- Your answers should be complete but concise.
- "Derive" means derive a result from fundamental principles.
- "Describe" means explain a result, but derivation is not necessary.

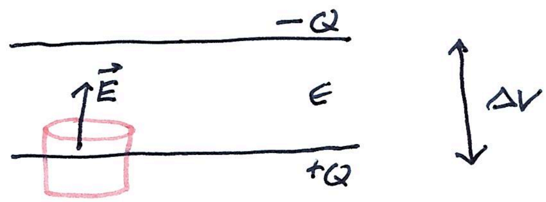
HINTS AND EQUATIONS:

Definition of capacitance = Q / ΔV

$$\nabla \times (\nabla \times \vec{F}) = \nabla (\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

$$\epsilon(\omega)/\epsilon_0 = 1 + \sum_j \frac{n_j e^2}{\epsilon_0 m} (\omega_j^2 - \omega^2 - i\omega \gamma_j)^{-1}$$

1. A parallel plate capacitor has area A and plate separation d.
The charges on the plates are +Q and −Q and the voltage difference is ΔV.
Between the plates there is a dielectric material with permittivity ϵ.
(a) Derive the capacitance.
(b) ±Q is free charge. Describe the *bound charge* and the *polarization* in this system.



(a) $\nabla \cdot \vec{D} = \rho \Rightarrow D_z \delta A = \delta q \quad \therefore D_z = \frac{Q}{A}$

$\Delta V = \int_0^d E_z \, dz = \frac{1}{\epsilon} \frac{Q}{A} d = \frac{Q}{C}$

capacitance $= \frac{\epsilon A}{d}$ (2)

(b) $\vec{D} = \epsilon_0 \vec{E} + \vec{P} \Rightarrow \vec{P} = (\epsilon - \epsilon_0) \frac{Q}{\epsilon A} \hat{e}_z$ (1)

$\nabla \cdot \vec{P} = 0 \Rightarrow \rho_{bound} = 0$ (1)

$\hat{n} \cdot \vec{P} = \pm (\epsilon - \epsilon_0) \frac{Q}{\epsilon A} = \pm \left(1 - \frac{\epsilon_0}{\epsilon}\right) \frac{Q}{A}$ (1)

$\hookrightarrow = \sigma_{bound}$

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2. A plane wave traveling in a simple linear medium with parameters ϵ and μ has this electric field,

$$\vec{E}(\vec{x},t) = E_0\, \hat{e}_x \sin(k z - \omega t)\ .$$

(a) Starting from Maxwell's equations, derive the magnetic field.

(b) Derive the relation between ω and k and the relation between B_0 and E_0 .

(c) Calculate the energy flux (units of watts / m^2).

(a) $$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \partial_z \\ E_x & 0 & 0 \end{vmatrix} = +\hat{y}\, \partial_z E_x$$

$$= \hat{y}\, k E_0 \cos(kz - \omega t)$$

$$\vec{B} = E_0 \frac{k}{\omega}\, \hat{y} \sin(kz-\omega t) = B_0\, \hat{y} \sin(kz - \omega t) \quad (1)$$

(b) $$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow B_0 = E_0\, k/\omega$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \Rightarrow \frac{k}{\mu} B_0 = \epsilon \omega E_0$$

$$\frac{k^2}{\mu \omega} = \epsilon \omega \Rightarrow k^2 = \mu\epsilon\, \omega^2 \quad (2)$$

$$\Rightarrow B_0 = \sqrt{\mu\epsilon}\, E_0$$

(c) $$\vec{S} = \vec{E} \times \vec{H} = \frac{E_0 B_0}{\mu}\, \hat{e}_z \sin^2(kz - \omega t) \quad (2)$$

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$$\frac{E_0 \sqrt{\mu\epsilon}\, E_0}{\mu} = \sqrt{\frac{\epsilon}{\mu}}\, E_0^2$$

3. For a free electron plasma there are no bound electrons, and the damping is negligible,
so the permittivity is simply

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 - \frac{\omega_p^2}{\omega^2} \quad \text{where} \quad \omega_p^2 = \frac{ne^2}{\epsilon_0 m}$$

(A) Show that electromagnetic waves with $\omega < \omega_p$ do not propagate in the plasma.
(B) Explain how the ionosphere affects radio communication. Give examples.
(C) Sketch a graph of the phase velocity verses ω, and label important points.

(A) $k^2 = \mu \epsilon \omega^2 = \mu \epsilon_0 (\omega^2 - \omega_p^2)$ (2)

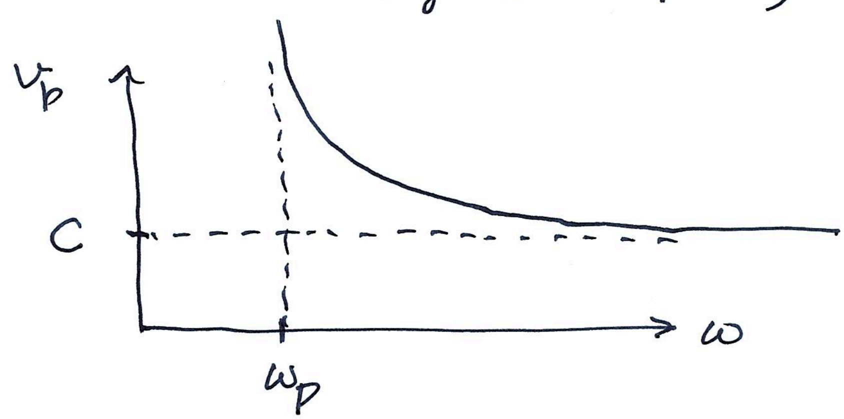
If $\omega < \omega_p$ then $k^2 < 0$; i.e., k is imaginary

$e^{ikx} = e^{-|k|x}$ does not propagate.

(B) If $\omega < \omega_p$ radio waves reflect from the ionosphere; e.g., AM radio (2)

If $\omega > \omega_p$ radio waves pass through the ionosphere; e.g., FM radio

(C) (2)



(6)

$$v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu\epsilon_0}} \frac{1}{\sqrt{1 - \omega_p^2/\omega^2}}$$

4. Consider an ideal waveguide—the walls are perfect conductors—with a square cross section. The dimensions are $\delta x = a$ and $\delta y = a$; waves propagate along the length, which is parallel to the the z axis. The lowest TM mode of propagation has $B_z = 0$ and

$$E_z(\vec{x},t) = E_0 \sin(\pi x/a)\sin(\pi y/a)\, e^{i(kz-\omega t)}.$$

$$\boxed{\left(\frac{\partial E_z}{\partial y}\right)_{y=0} = \frac{\pi}{a} E_0 \sin\left(\frac{\pi x}{a}\right) e^{i(kz-\omega t)}}$$

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(a) Describe the relation between k and ω.

(b) What is the minimum frequency for propagation?

(c) ~~(b)~~ Calculate the charge density on the walls.

(a) $\vec{E}$ obeys the wave equation : $\left(\nabla^2 - \mu\epsilon \frac{\partial^2}{\partial t^2}\right) E_z = 0$

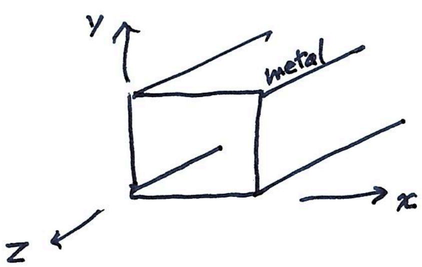
Therefore
$$-\frac{\pi^2}{a^2} - \frac{\pi^2}{a^2} - k^2 + \mu\epsilon\omega^2 = 0$$

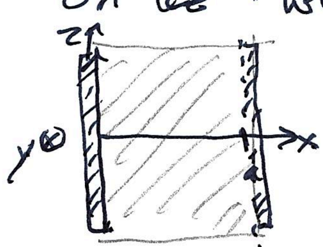
$$\mu\epsilon\,\omega^2 = \frac{2\pi^2}{a^2} + k^2 .$$ ← (2 points)

(b) The cut off frequency is $\omega_c = \frac{1}{\sqrt{\mu\epsilon}} \frac{\sqrt{2}\,\pi}{a}$. ← (2 points)
(k = 0)

← (4 points)

(c)





On the "wall" at $y = 0$ (the "floor")
$\hat{x}$ and $\hat{z}$ are tangential
$\hat{y}$ is normal to the wall
Also, $y = 0$, $x \in (0, a)$

∴ The $\hat{B}$ boundary conditions are $B_y = 0$ and $B_x = \mu(\vec{K} \times \hat{n})_x$
$= \mu(-\hat{x}K_z + \hat{z}K_x)$ $\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ K_x & 0 & K_z \\ 0 & 1 & 0 \end{vmatrix}$
$(\because K_x = 0)$

And the $\hat{E}$ boundary conditions are $E_x = E_z = 0$ and

$E_y = \frac{1}{\epsilon} D_y = \frac{1}{\epsilon}\Sigma$ where Σ = charge density

$$\boxed{\Sigma(x,0,z) = \epsilon E_y(x,0,z)}$$

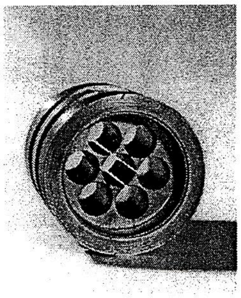
Now, what is E_y at $y = 0$?

$(\nabla \times \vec{E})_x = -\frac{\partial \vec{B}_x}{\partial t} = i\omega B_x = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & ik \\ E_x & E_y & E_z \end{vmatrix}_{(x)} = \frac{\partial E_z}{\partial y} - ikE_y$

$(\nabla \times \vec{H})_y = \frac{\partial \vec{D}_y}{\partial t} = -i\omega D_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & ik \\ H_x & H_y & 0 \end{vmatrix}_{(y)} = ikH_x$

$$\boxed{\begin{aligned} i\omega B_x &= \frac{\partial E_z}{\partial y} - ikE_y \\ -i\omega D_y &= ikH_x \\ &\text{∴ SOLVE FOR } E_y! \end{aligned}}$$

5. What is shown in the picture? How does it work?



- Anode of a cavity magnetron
- Electrons travel on spiral orbits
- the electron current excites cavity resonances (microwaves) are generated

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6. Using text, pictures and equations, explain why the sky is blue on a clear sunny day.

— cross section $\propto \omega^4$

— scattered light is more intense for high frequencies (blue or violet)

— observer sees a spectrum peaked at high frequencies (blue)

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ω^4 scattering spectrum picture