Lecture 1–1

Elementary treatment of electrostatics with ponderable materials

Jackson: Sections 4.1 4.2 4.3

Recall ...

... the field equations of microscopic electrostatics ,

$$\nabla \cdot \vec{E} = \rho / \epsilon_0 \quad \text{and} \quad \nabla \times \vec{E} = 0$$

Here $\rho(\vec{x})$ is the total charge density that creates the field $\vec{E}(\vec{x})$; and these functions are independent of time .

In principle the solution is

$$\vec{E}(\vec{x}) = \int \frac{(\vec{x} - \vec{x}')}{4 \pi \epsilon_0 \, | \, \vec{x} - \vec{x}' \, |^3} \rho(\vec{x}') \, d^3x'$$

This is OK if $\rho(\vec{x})$ consists of a few isolated charges or simple charge densities. But what if ρ includes charges from a macroscopic sample of matter?

¶ All electric charge comes from electrons and nuclei. The number of charged particles is terribly large, like Avogadro's number! We cannot handle a calculation that treats each particle individually.

Instead, we average over small volumes ΔV that contain large numbers of particles. It's a subtle limiting process,

$$\text{molecule} \ll \Delta V \ll V,$$

which is possible because the size of a molecule is so tiny.

The macroscopic density and the macroscopic field

$$\rho_{\text{macro}}(\vec{x}) = \langle \rho(\vec{x}) \rangle = \frac{1}{\Delta V} \int_{\Delta V} \rho(\vec{x}+\vec{\xi})\, d^3\xi$$

$$\vec{E}_{\text{macro}}(\vec{x}) = \langle \vec{E}(\vec{x}) \rangle = \frac{1}{\Delta V} \int_{\Delta V} \vec{E}(\vec{x}+\vec{\xi})\, d^3\xi$$

Draw a picture!

Free charge and bound charge

Next, we divide $\langle \rho(\vec{x}) \rangle$ into two parts,

$$\langle\rho(\vec{x})\rangle = \rho_{free}(\vec{x}) + \rho_{bound}(\vec{x})$$

Make sure that you understand the meanings of these two charge densities!

For insulators ("dielectrics"):
ρ_{free} is the "excess charge" added to the system;
ρ_{bound} is the charge that belongs to the molecules in the material.

For metals ("conductors"):
ρ_{free} is the "excess charge" plus con-duction electrons;
ρ_{bound} is the charge that belongs to ions in the material.

Derivation of ρ_{bound}

Suppose the molecular density at the point $\vec{x}$ is $N(\vec{x})$. You might think that we could write

$$\rho_{bound}(\vec{x}) = N(\vec{x}) \langle\rho_{molecule}(\vec{x})\rangle$$

where

$$\rho_{mol.}(\vec{x}) = \sum_{i \in mol.} e_i \, \delta^3(\vec{x}-\vec{x}_i) \, ;$$

but that is certainly wrong, at least for dielectrics, because $\langle\rho_{mol.}\rangle = 0$!
Molecules are electrically neutral; $\sum_i e_i = 0$.

$\rho_{mol.}(\vec{x})$ is not zero, but $\langle\rho_{mol.}(\vec{x})\rangle$ is zero; the averaging over ΔV gives 0. (Metals are a little different.)

—— Then what is ρ_{bound}? ——

For a dielectric, the answer is

$$\rho_{\text{bound}}(\vec{x}') = -\nabla' \cdot (N(\vec{x}') \langle \vec{p}_{\text{mol},x'} \rangle)$$

where $\vec{p}_{\text{mol},x'}$ = the electric dipole moment of a molecule centered at $\vec{x}'$; and $\langle\,\rangle$ means the average over ΔV.

$$\vec{p}_{\text{mol},x'} = \int \vec{\xi}\, \rho_{\text{mol},x'}(\vec{\xi})\, d^3\xi$$

The above equation introduces a vector function $\vec{P}(\vec{x}')$ called the *polarization,* defined by

$$\vec{P}(\vec{x}') = N(\vec{x}') \langle \vec{p}_{\text{mol},x'} \rangle$$
$$= \textit{the dipole moment density.}$$

Theorem

$$\rho_{\text{bound}}(\vec{x}) = -\nabla \cdot \vec{P}(\vec{x})$$

Proof ...

Start with the electric field created by the charges in a single molecule at $\vec{x}'$; call that $\vec{e}_1(\vec{x})$;   let $\vec{r} = \vec{x} - \vec{x}'$;

$$\vec{e}_1(\vec{x}) = \int \frac{(\vec{r} - \vec{\xi})}{4\pi\epsilon_0 \,|\vec{r} - \vec{\xi}|^3} \rho_{\text{mol}}(\vec{\xi})\, d^3\xi$$

$$= -\nabla \int \frac{\rho_{\text{mol}}(\vec{\xi})\, d^3\xi}{4\pi\epsilon_0 \,|\vec{r} - \vec{\xi}|}$$

Here the integration is over the tiny interior of the molecule, ie, where $\rho_{\text{mol}} \neq 0$.

We will eventually average over ΔV which is large compared to the molecule. So we can approximate $\vec{e}_1$ by taking $r \gg \xi$;

$$\frac{1}{\left|\vec{r} - \vec{\xi}\right|} \simeq \frac{1}{r} + \frac{\hat{r} \cdot \vec{\xi}}{r^2} + O\left(\frac{\xi^2}{r^3}\right)$$

$$\therefore \Phi_1(\vec{x}) \simeq \frac{1}{4\pi\epsilon_0}\left\{\frac{q_1}{r} + \frac{\hat{r}\cdot\vec{p}_1}{r^2} + \text{neglect}\right\}$$

$$q_1 = \int \rho_{mol}(\vec{\xi})\, d^3\xi = \Sigma\, e_i = 0$$

$$\vec{p}_1 = \int \rho_{mol}(\vec{\xi})\,\vec{\xi}\, d^3\xi = \text{dipole moment}$$

$$= \Sigma\, e_i\, \vec{\xi}_i$$

$$\Phi_1(\vec{x}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot\vec{p}_1}{r^2}$$

where $\vec{r} = \vec{x} - \vec{x}'$.

Now replace $\vec{p}_1$ by $\langle\vec{p}_1\rangle$; and multiply by $N(\vec{x}')$; and integrate over $\vec{x}'$; the result is the macroscopic potential at $\vec{x}$ …

$$\Phi(\vec{x})=\int \frac{(\vec{x} - \vec{x}')\cdot \vec{P}(\vec{x}')}{4\pi\epsilon_0 \mid \vec{x} - \vec{x}' \mid^3} \, d^3x'$$

The result is obvious, isn't it?

$\vec{P}(\vec{x}')\,\delta V'$ = the net dipole moment of the volume $\delta V'$.

Now we need a bit of calculus to derive $\rho_{\text{bound}}(\vec{x})$.

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{(\vec{x}-\vec{x}')\cdot\vec{P}(\vec{x}')}{\mid \vec{x}-\vec{x}'\mid^3}\, d^3x'$$

$$= k \int \vec{P}(\vec{x}')\cdot \nabla' \frac{1}{\mid\vec{x}-\vec{x}'\mid}\, d^3x'$$

$$= k \int \left\{ \nabla'\cdot\left(\frac{\vec{P}(\vec{x}')}{\mid\vec{x}-\vec{x}'\mid}\right) - \frac{\nabla'\cdot\vec{P}(\vec{x}')}{(\vec{x}-\vec{x}')} \right\}\, d^3x'$$

$\sqrt{}$ use Gauss's theorem

$$= \int_S \frac{\hat{n}'\cdot\vec{P}(\vec{x}')}{\mid\vec{x}-\vec{x}'\mid}\, dA'$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{\sigma_{\text{bound}}\,\hat{n}'\cdot d\vec{A}'}{\mid\vec{x}-\vec{x}'\mid} + \frac{1}{4\pi\epsilon_0}\int \frac{\rho_{\text{bound}}(\vec{x}')}{(\vec{x}-\vec{v})}$$

where

$$\rho_{\text{bound}}(\vec{x}') = -\nabla'\cdot\vec{P}(\vec{x}') \quad /QED$$

$$\sigma_{\text{bound}}(\vec{x}') = \hat{n}'\cdot\vec{P}(\vec{x}')$$

Result

Result

If the molecules are polarized, and the

polarization function is $\vec{P}(\vec{x})$, then:

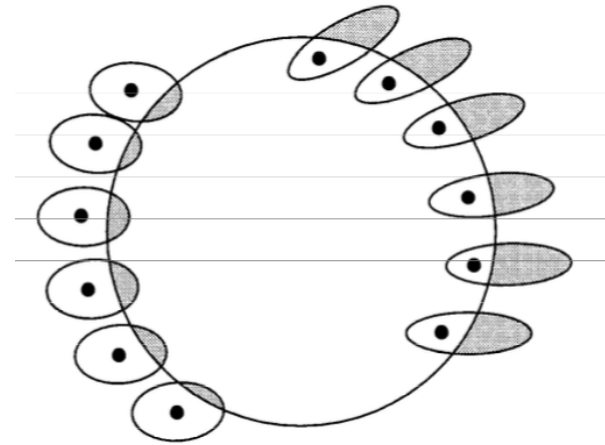■ there is a *surface charge density* σ_{bound} on the boundary of V,

$$\sigma_{bound} = \hat{n} \cdot \vec{P}$$

■ and there is a *volume charge density* ρ_{bound} inside the volume,

$$\rho_{bound} = -\nabla \cdot \vec{P}$$

Comments

Study this figure from Jackson

Polarization does not necessarily produce a volume charge density. But **divergence** of polarization does produce a volume charge density.

The macroscopic field equations in electrostatics

Start with the microscopic field equations,

$$\nabla \cdot \vec{E} = \rho / \epsilon_0 \ \text{ and } \ \nabla \times \vec{E} = 0 \,.$$

Now derive the macroscopic field equations.

■ Average over small volume elements ΔV throughout the medium.

■ Replace $\langle \rho(\vec{x}) \rangle$ by

$$\rho_{\text{free}}(\vec{x}) + \rho_{\text{bound}}(\vec{x}) = \rho_{\text{free}}(\vec{x}) - \nabla \cdot \vec{P}(\vec{x}).$$

■ Replace $\langle \vec{E}(\vec{x}) \rangle$ by $\vec{E}(\vec{x})$.

the average of the microscopic field is the macroscopic field; [be careful: we're using the same symbol for both!]
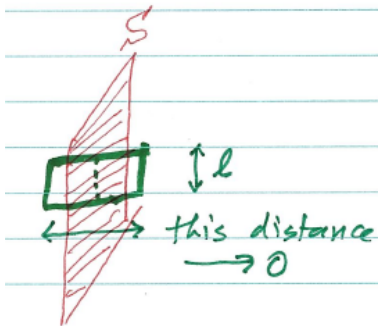
■ Define the *displacement field* $\vec{D}(\vec{x})$ by

$$\vec{D} = e_0 \, \vec{E} + \vec{P}$$

Thus,

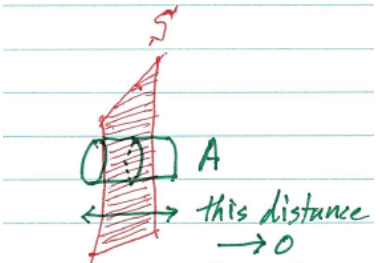$$\nabla \cdot \vec{D} = \rho_{\text{free}} \quad \text{and} \quad \nabla \times \vec{E} = 0$$

Boundary conditions for $\vec{E}(\vec{x})$ and $\vec{D}(\vec{x})$

$\updownarrow \ell$

this distance $\longrightarrow 0$

$E_{tangential}$ is continuous.

$$\nabla \times \vec{E} = 0 \implies \oint \vec{E} \cdot \vec{dl} = 0$$

$$E_< \, l - E_> \, l = 0$$

A

this distance $\longrightarrow 0$

$$\Delta D_{normal} = \sigma_{free} \, .$$

$$\nabla \cdot \vec{D} = \rho_{free} \implies$$

$$\oint \vec{D} \cdot \vec{dA} = Q_{free}$$

$$D_< \, A - D_> \, A = \sigma_{free \, A}$$