start

Lecture 1–2 (Wed Sept 4)
Boundary Value Problems with Dielectrics

Jackson: Section 4.4

The equations of macroscopic electrostatics ...

$$\nabla \times \vec{E} = 0 \quad \text{and} \quad \nabla \cdot \vec{D} = \rho_{\text{free}}$$

where $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

$\vec{P}(\vec{x})$ = polarization in the medium
= dipole moment density

B*oundary conditions;* at any surface,

$E_{\text{tangential}}$ is continuous;

D_{normal} is continuous,

or $\Delta D_n = \sigma_{\text{free}}$

Constitutive Equations

$\longmapsto$ relations between $\vec{E}$ and $\vec{P}$
(or, $\vec{E}$ and $\vec{D}$ where $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$.)

The simplest model for polarization (a linear model)

Recall, $\vec{P}(\vec{x}) = N(\vec{x}) \langle \vec{p}_{\text{mol},x} \rangle$.

Assume $\vec{P}(\vec{x})$ is proportional to $\vec{E}(\vec{x})$.

How can this fail? *(Actually, it can fail!)*

Assuming the material is uniform and isotropic, we write

$$\vec{P}(\vec{x}) = \epsilon_0 \chi_e \vec{E}(\vec{x})$$

$\chi_e \equiv$ electric susceptibility

What are the units of χ_e ?

Or,

$$\vec{D}(\vec{x}) = \epsilon_0\, \vec{E}(\vec{x}) + \vec{P}(\vec{x}) = \epsilon_0\, (1 + \chi_e)\, \vec{E}(\vec{x})$$

$$\vec{D}(\vec{x}) = \epsilon\, \vec{E}(\vec{x}) \quad \text{where} \quad \epsilon = \epsilon_0\, (1 + \chi_e)$$

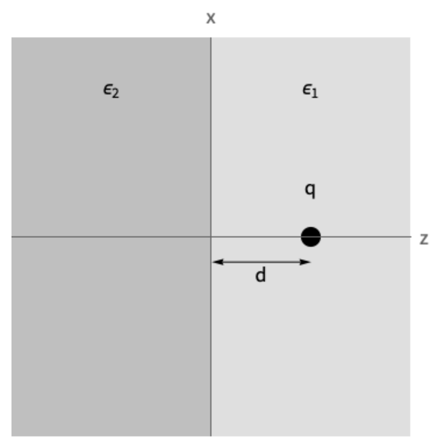
$$\epsilon = \text{electric permittivity}$$

Later we'll try to derive χ_e *from theory, but for now just take it as a lab measurement.*

Boundary Value Problems

Example 1

First, draw a picture to define the problem.



The problem is to calculate the electric field $\vec{E}(\vec{x})$. The field equations are

$$\nabla \cdot \vec{D} = \rho_{free}(\vec{x}) = q\, \delta^3(\vec{x} - \hat{e}_z\, d)$$

$$\nabla \times \vec{E} = 0$$

... with these constitutive equations

$$\vec{D} = \epsilon_1 \vec{E} \text{ in region R1 } (z > 0)$$

$$\vec{D} = \epsilon_2 \vec{E} \text{ in region R2 } (z < 0)$$

Solution

First, because $\nabla \times \vec{E} = 0$ we can write
$\vec{E}(\vec{x}) = -\nabla\Phi(\vec{x})$;
now the problem is to determine $\Phi(\vec{x})$.
Next, $\nabla \bullet \vec{D} = \rho_{\text{free}}$ where $\vec{D} = \epsilon\, \vec{E} = -\,\epsilon\, \nabla\Phi$.

$$\begin{cases} \epsilon_1 \nabla^2 \Phi = -q\, \delta^3\left(\vec{x} - \hat{e}_z\, d\right) & \text{in region R1 } (z > 0) \\ \epsilon_2 \nabla^2 \Phi = 0 & \text{in region R2 } (z < 0) \end{cases}$$

The problem has cylindrical symmetry, but we won't use that to find the solution. Instead we will use the familiar trick called the ***method of images.***

Region R1 For z > 0,

$$\epsilon_1 \Phi(\vec{x}) = \frac{q}{4\pi \left| \vec{x} - \hat{e}_z d \right|} + \Phi'(x)$$

where $\nabla^2\Phi' = 0$. This does not mean that Φ' = 0! We can "guess" that the form of Φ' is the potential due to a fictitious "image" charge q' at $-\hat{e}_z$d (*over in region R2*). So

$$4\pi\, \epsilon_1 \Phi(\vec{x}) =$$

$$= \frac{q}{\left| \vec{x} - \hat{e}_z d \right|} + \frac{q'}{\left| \vec{x} + \hat{e}_z d \right|}, \quad \text{for } \vec{x} \in R1$$

Region R2 For z < 0,

$\nabla^2\Phi = 0$. This does not mean that Φ = 0! We can "guess" that the form of Φ = the potential due to a fictitious "image" charge q" at $+\hat{e}_z$d (*over in region R1*). So

$$4\pi\, \epsilon_2 \Phi(\vec{x}) =$$

$$= \frac{q''}{\left| \vec{x} - \hat{e}_z d \right|} \quad \text{for } \vec{x} \in R2$$

So, now we have $\nabla \bullet \vec{D} = \rho_{\text{free}}$, guaranteed! But what are q' and q"?

Apply the boundary conditions.

■ $D_{\text{norm.}}$ is continuous at z = 0 (no free charge on the surface) ⟹

$$\epsilon_2 \frac{\partial \Phi}{\partial z}(x,y,0-) = \epsilon_1 \frac{\partial \Phi}{\partial z}(x,y,0+)$$

$$\therefore\ q'' = q - q'$$

■ $E_{\text{tang.}}$ is continuous at z = 0 ⟹

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In[*]:= Style[" ∂Φ/∂x (x,y,0-) = ∂Φ/∂x (x,y,0+)", ff]
Style[" ∴ q''/ϵ2 = (q + q')/ϵ1 ", ff]
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$$\frac{\partial \Phi}{\partial x}(x,y,0-) = \frac{\partial \Phi}{\partial x}(x,y,0+)$$

$$\therefore\ \frac{q''}{\epsilon_2} = \frac{q + q'}{\epsilon_1}$$

Now solve for q’ and q” .


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In[*]:= Style["q' = - (ϵ2 - ϵ1)/(ϵ2 + ϵ1) q  and  q'' = 2 ϵ2/(ϵ2 + ϵ1) q", ff]
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$$q' = -\frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}\, q \quad \text{and} \quad q'' = \frac{2\,\epsilon_2}{\epsilon_2 + \epsilon_1}\, q$$

See Figure 4.5.

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In[*]:= Show[figex1, ImageSize → 480]
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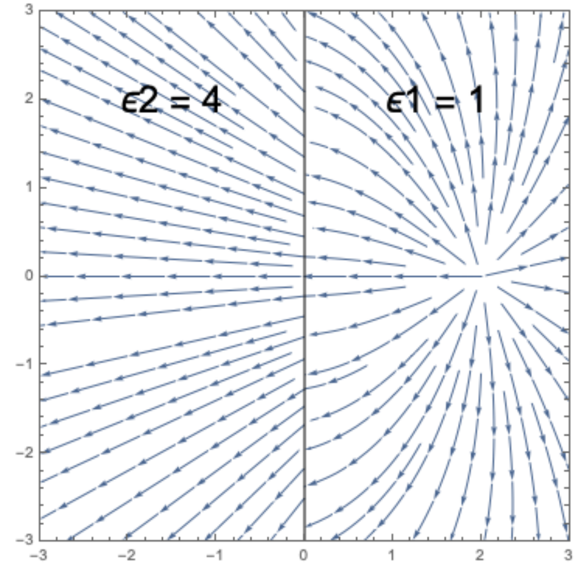


What is plotted here?

Is it $\vec{E}$ or $\vec{D}$?

Exercises.

■ Determine the surface charge density on the boundary surface.

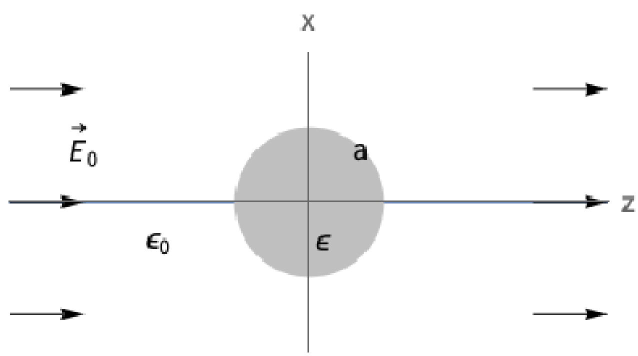
$$\sigma_{micro} = E_{2\,n} - E_{1\,n}$$

■ Determine the force on q. Is it attractive or repulsive?

Example 2

A dielectric sphere is placed in a uniform external electric field. Calculate $\vec{E}(\vec{x})$.

First, draw a picture to define the problem.



$$\Phi_{in} = \sum_{l=0}^{\infty} A_l\, r^l\, P_l(\cos\theta) \quad (r<a)$$

$$\Phi_{out} = \sum_{l=0}^{\infty} (B_l\, r^l + C_l r^{-(l+1)})\, P_l(\cos\theta) \quad (r>a)$$

Far from the sphere, i.e., as r $\longrightarrow \infty$, Φ must approach the potential for a uniform field $= -E_0 z = -E_0 r \cos\theta$. Therefore $B_1 = -E_0$ and $B_l = 0$ for $l \neq 1$.

Now we have $\nabla^2\Phi = 0$, guaranteed.

Next, apply the boundary conditions.

$E_{tang.}$ is continuous at r = a ; implies

$$E_{in,\theta}(a,\theta) = E_{out,\theta}(a,\theta)$$

$$-\frac{1}{a}\left[\frac{\partial\Phi_{in}}{\partial\theta}\right]_{r=a} = -\frac{1}{a}\left[\frac{\partial\Phi_{out}}{\partial\theta}\right]_{r=a}$$

$$\therefore\ A_1 = -E_0 + \frac{C_1}{a^3} \quad \text{for the case } l = 1$$

$D_{norm.}$ is continuous $\Longrightarrow$

$$\epsilon\, E_{in,r}(a,\theta) = \epsilon_0\, E_{in,r}(a,\theta)$$

$$-\epsilon\left[\frac{\partial\Phi_{in}}{\partial r}\right]_{r=a} = -\epsilon_0\left[\frac{\partial\Phi_{out}}{\partial r}\right]_{r=a}$$

$$\therefore\ \epsilon A_1 = -\epsilon_0 E_0 - \frac{2\epsilon_0 C_1}{a^3} \quad \text{for } l = 1$$

Exercise : Show that $A_l = 0$ and $C_l = 0$ for $l \neq 1$; this is sort of obvious.

```
(* Calculation *)
eqs = {A1 == -E0 + C1 / a ^ 3, ∈ * A1 == -∈0 * E0 - 2 ∈0 C1 / a ^ 3};
A1 /. Part[Solve[eqs, {A1, C1}] // Simplify, 1]
C1 /. Part[Solve[eqs, {A1, C1}] // Simplify, 1]
```

$$-\frac{3\, E0\, \epsilon0}{\epsilon + 2\,\epsilon0}$$

$$\frac{a^3\, E0\, (\epsilon - \epsilon0)}{\epsilon + 2\,\epsilon0}$$

Thus, solving for A_1 and C_1 ,

$$A_1 = -\frac{3\,\epsilon_0}{\epsilon + 2\,\epsilon_0}\,E_0 \;\;\text{and}\;\; C_1 = \frac{\epsilon - \epsilon_0}{\epsilon + 2\,\epsilon_0}\,a^3\,E_0$$

■ The potential inside the sphere corresponds to a constant electric field,

$$\Phi_{in} = -\frac{3\,\epsilon_0}{\epsilon + 2\,\epsilon_0} E_0\,r\cos\theta$$

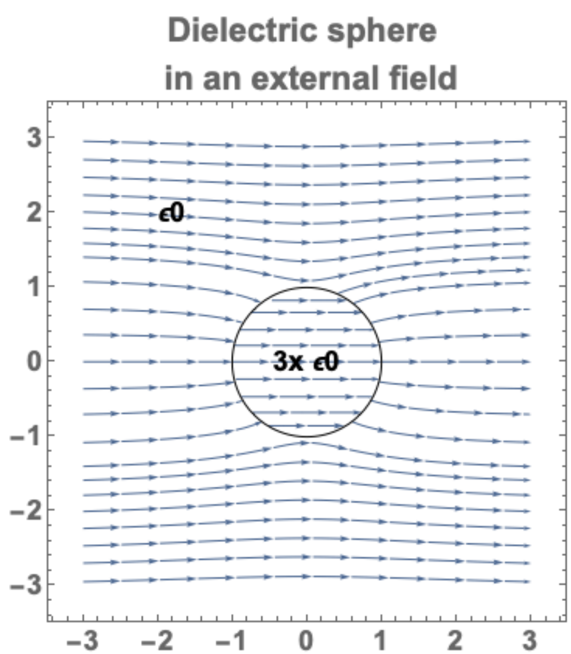
$$\therefore \vec{E}_{in} = \frac{3\,\epsilon_0}{\epsilon + 2\,\epsilon_0}\,E_0\,\hat{e}_z$$

The potential outside the sphere corresponds to the applied electric field plus the field of an electric dipole,

$$\Phi_{out} = -E_0\,r\cos\theta + \frac{\epsilon - \epsilon_0}{\epsilon + 2\,\epsilon_0}\,\frac{E_0\,a^3}{r^2}\cos\theta\,;$$

$$\therefore \vec{E}_{out} = E_0\,\hat{e}_z - \nabla\left(\frac{\vec{x}\cdot\vec{p}}{4\,\pi\epsilon_0\,r^3}\right)$$

$$\vec{p} = \left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\,\epsilon_0}\right) a^3\,E_0\,\hat{e}_z$$

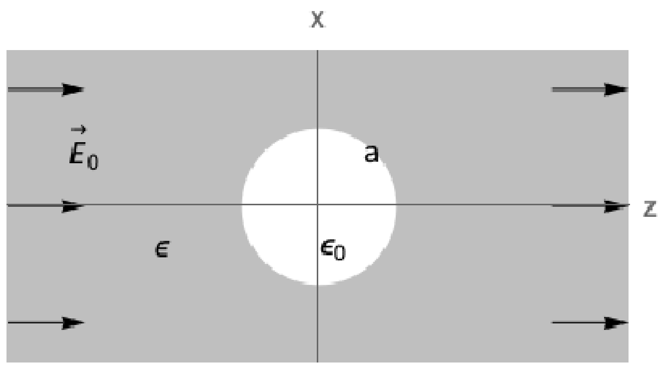


plotting the $\vec{E}$ field lines in the xz-plane, for $\epsilon = 3\,\epsilon_0$.

Example 3

A spherical *cavity* in a dielectric medium with an applied electric field.

First, draw a picture to define the problem.



This is the same as Example 2, except for the replacement $\epsilon/\epsilon_0 \longrightarrow \epsilon_0/\epsilon$.

So comparing to the final results from Example 2,

$$\vec{E}_{\text{in the cavity}} = \frac{3\,\epsilon}{2\,\epsilon + \epsilon_0}\,\vec{E}_0$$

i.e., the electric field is stronger in the cavity; and,

$$\vec{p}_{\text{outside}} = \frac{\epsilon - \epsilon_0}{2\,\epsilon + \epsilon_0}\,a^3\,E_0\,(-\,\hat{e}_z)$$

i.e., the dipole moment *due to the cavity* points antiparallel to the applied field (+ and – charges are reversed!).