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Lecture 1–5 {Wed, Sept. 11}

Electrostatic Energy in Dielectric Media

Jackson Section 4.7

ENERGY IN ELECTROSTATICS

Microscopic electrostatics

Consider a system with N charges (draw a picture!).

What is the energy?
Start with 2 charges.
The work required to assemble 2 charges =

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$$W^{(2)} = \frac{q_1 q_2}{4\pi\epsilon_0 \left| \vec{x}_1 - \vec{x}_2 \right|};$$

$$\text{sim. } W^{(N)} = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{q_i q_j}{4\pi\epsilon_0 \left| \vec{x}_i - \vec{x}_j \right|} \text{ with } i > j\,;$$

$$\text{or, } W^{(N)} = \frac{1}{2} \sum_i \sum_j \frac{q_i q_j}{4\pi\epsilon_0 \left| \vec{x}_i - \vec{x}_j \right|} \text{ with } i \neq j\,;$$

$$= \frac{1}{2} \sum_i q_i \, \Phi^{(N-1)}(\vec{x}_i)$$

In terms of charge density,

$$\rho(\vec{x}) = \sum_{i=1}^{N} q_i\, \delta^3(\vec{x} - \vec{x}_i) \longrightarrow \sum_{i=1}^{N} q_i\, f(\vec{x} - \vec{x}_i)$$

better replace point charges
by distributed charges

Now consider

$$U = \frac{1}{2} \int d^3x\, \rho(\vec{x})\, \Phi(\vec{x}) \quad \text{where} \quad \Phi(\vec{x}) = \int \frac{\rho(x')\, d^3x'}{4\pi\epsilon_0 \left| \vec{x} - \vec{x}' \right|}$$

= self energies + interaction energies

$$\text{self energy } i = \frac{q_i^2}{2} \int \frac{f\left(\vec{x}\right) f\left(\vec{x}'\right) d^3x\, d^3x'}{4\pi\epsilon_0 \left| \vec{x} - \vec{x}' \right|}$$

(infinite for a point charge!)

So, for microscopic electrostatics,

$$U = \frac{1}{2} \int d^3x\, \rho(\vec{x})\, \Phi(\vec{x}) \qquad (1)$$

Is this equation valid for macroscopic electrostatics? Not obvious because then ρ and Φ

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mean something different from ρ_{micro} and Φ_{micro}. We'll find that equation (1) is only true for linear materials.

Energy in a macroscopic dielectric material

$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$; but do not assume linear .

Let W = the energy required to assemble a *free charge density* $\rho(\vec{x})$, **AND** to produce the final polarization $\vec{P}(\vec{x})$.

Differential changes: Suppose we already have free charge ρ and potential Φ. Now we add free charge $\delta\rho$

$$\Longrightarrow \delta W = \int \delta\rho(\vec{x})\, \Phi(\vec{x})\, d^3x \,.$$

Calculations ...

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$$\nabla \cdot \vec{D} = \rho \quad \Longrightarrow \quad \delta\rho = \nabla \cdot \delta\vec{D}$$

$$\therefore \delta W = \int (\nabla \cdot \delta\vec{D})\, \Phi \; d^3x$$

$$= \int (-)\, \delta\vec{D} \cdot \nabla\Phi \, d^3x = \int \delta\vec{D} \cdot \vec{E} \, d^3x$$

using integration by parts

$\Longrightarrow$

$$W = \int d^3x \int_0^{D} \vec{E} \cdot d\vec{D} \qquad (2)$$

W is the energy of the dielectric system.

Special case: a linear medium

OK, now assume $\vec{D} = \epsilon\,\vec{E}$.

Then $\delta(\vec{E} \bullet \vec{D}) = \delta\vec{E} \bullet \vec{D} + \vec{E} \bullet \delta\vec{D}$

$= 2\,\epsilon\,\delta\vec{E} \bullet \vec{E} = 2\,\vec{E} \bullet \delta\vec{D}$

and so

$$W = \int d^3x \int_0^D \frac{1}{2}\, d(\vec{E}\bullet\vec{D})$$

$$= \frac{1}{2} \int \vec{E} \bullet \vec{D}\, d^3x \qquad (3)$$

for a linear medium.

We could rewrite (3) as

$$W = \frac{1}{2} \int \rho(\vec{x})\, \Phi(\vec{x})\, d^3x \qquad (4)$$

because $\vec{E} = -\,\text{grad}\,\Phi$ and div $\vec{D} = \rho$.

However, (4) and (1) are not identical because in (1) $\rho = \rho_{\text{micro}}$ and in (4) $\rho = \rho_{\text{free}}$!

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Example.

Assume:

(i) all free charges are fixed;

(2) the initial medium is free space;

(3) the medium changes by adding a linear dielectric.

Calculate W.

$$W_0 = \frac{1}{2} \int \vec{E}_0 \bullet \vec{D}_0\, d^3x \quad \text{where } \vec{D}_0 = \epsilon_0\, \vec{E}_0$$

$$W = \frac{1}{2} \int \vec{E} \bullet \vec{D}\, d^3x \quad \text{where } \vec{D} = \epsilon\, \vec{E}$$

$$\Delta W = \frac{1}{2} \int (\, \vec{E} \bullet \vec{D} - \vec{E}_0 \bullet \vec{D}_0\,)\, d^3x$$

$$W = \frac{1}{2}\int\left(\vec{E}\cdot\vec{D} - \vec{E}_0\cdot\vec{D}_0\right)d^3x$$

$$= \frac{1}{2}\int\left(E\cdot D_0 - D\cdot E_0 + (\vec{E}+\vec{E}_0)\cdot(\vec{D}-\vec{D}_0)\right)d^3x$$

integral is 0

$$\int -\nabla\Phi\cdot(D-D_0)\,d^3x = \int \Phi\,\nabla\cdot(\vec{D}-\vec{D}_0)\,d^3x$$

$$\rho-\rho_0=0$$

$$= \frac{1}{2}\int\left(\vec{E}\cdot\vec{D}_0 - \vec{D}\cdot\vec{E}_0\right)d^3x$$

Result: $\Delta W = \dfrac{1}{2}\displaystyle\int_{\Omega_1}(\vec{E}\cdot\vec{D}_0 - \vec{D}\cdot\vec{E}_0)\,d^3x$

where Ω_1 = the volume of dielectric (because outside Ω_1 the integrand is 0).

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Again, a linear dielectric has

$$W = \frac{1}{2}\int_{\Omega_1}(\epsilon_0 - \epsilon)\,\vec{E}\cdot\vec{E}_0\,d^3x$$

$$W = -\frac{1}{2}\int_{\Omega 1}\vec{P}\cdot\vec{E}_0\,d^3x$$

Note: Since $\epsilon > \epsilon_0$, the dielectric object will be pulled toward the the region of stronger electric field.

Solution to this kind of problem:

The energy density of a linear dielectric placed in a system with fixed free charge is

$$u = -\frac{1}{2}\vec{P}\cdot\vec{E}_0\ .$$

The force on a dielectric

Example 1: fixed sources

Start with electric field $\vec{E}_0$ in free space, *with fixed free charges*. Now add a dielectric object to the system. Calculate the force on the object.

Make a generalized displacement $\delta\xi$; e.g., translation or rotation. The generalized force is

$$F_\xi = -\left(\frac{\partial W}{\partial \xi}\right)_Q$$

(sub Q means free charges are kept fixed)

Homework problem

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Example 2: fixed potentials

A related problem ...

Start with electric field E_0 in free space, with *fixed potentials* on various conductors of the system. Now add a dielectric object to the system. Calculate the force on the object.

The previous equation is not correct, because now there is another source of energy—the batteries that are keeping the potentials constant. Free charge will move in the conductors to which the electrodes are attached.

A differential change in the dielectric now produces both $\delta\rho$ and $\delta\Phi$. The differential change of energy is

$$\delta W = \frac{1}{2}\int (\rho\,\delta\Phi + \Phi\,\delta\rho)\,d^3x$$

Think of it as two steps:
(i) first disconnect the batteries and change the dielectric; $\delta\rho_{\text{free}} = 0$ so

$$\delta W_1 = \frac{1}{2} \int \rho \, \delta\Phi_1 \, d^3x$$

(ii) then reconnect the batteries;

Charge $\delta\rho_2 \, (\vec{x})$ flows to or from the batteries, to restore the original potentials; so ...

$$\delta\Phi_2 = - \delta\Phi_1$$

$$\therefore \delta W_2 = \tfrac{1}{2} \int (\rho \, \delta\Phi_2 + \Phi \, \delta\rho_2) \, d^3x = -2 \, \delta W_1$$

(Note: $\int \rho \, \delta\Phi_2 = \int \Phi \, \delta\rho_2$; and $\delta\Phi_2 = - \delta\Phi_1$)

The combined change of energy is

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$$\delta W = \delta W_1 + \delta W_2 = - \delta W_1 = - \tfrac{1}{2} \int \rho \, \delta\Phi_1 \, d^3x$$

$$(\delta W)_V = - (\delta W)_Q$$

$$\therefore \vec{F} = + (\nabla W)_V \qquad \text{(a surprising sign!)}$$

The most familiar example is a capacitor with a moveable dielectric slab.