Topic#2

Magnetostatics with Macroscopic Media

Jackson: Chapter 5, Sections 8-12

PHY 841 : magnetostatics in "empty space"; Jackson Sections 5.1 - 5.7

PHY 842 : magnetostatics in the presence of macroscopic media;

diamagnetic, paramagnetic, and ferromagnetic media.

Lecture 2-1 {Fri, 9-13}

Review of Microscopic Magnetostatics

Jackson Sections 5.1 – 5.6. (= ***review***)

The microscopic field equations of Magnetostatics

$$\text{div}\,\vec{B} = 0 \quad \text{and} \quad \text{curl}\,\vec{B} = \mu_0\,\vec{J}$$

$$\text{where} \quad \text{div}\,\vec{J} = 0$$

Here $\vec{B}(\vec{x})$ and $\vec{J}(\vec{x})$ are independent of t.

Section 5–1
Introduction and Definitions

$\vec{N} = \vec{\mu} \times \vec{B}$	$\vec{B}$ = magnetic induction
$\partial\rho/\partial t + \nabla \cdot \vec{J} = 0$	magnetostatics, $\nabla \cdot \vec{J} = 0$

Section 5–2
Biot and Savart Law

$d\vec{B} = \frac{\mu_0 I \, d\vec{l} \times \vec{r}}{4\pi r^3}$	constant current
$\vec{B} = \frac{\mu_0 q \vec{v} \times \vec{r}}{4\pi r^3}$	nonrelativistic particle

For a long straight wire,

$$\vec{B}(\vec{x}) = \frac{\mu_0 I}{2\pi R} \hat{e}_\phi$$

Section 5–3
Ampere’s Law

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}\,') \times (\vec{x} - \vec{x}\,')}{|\vec{x} - \vec{x}\,'|^3} d^3x'$$

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \nabla \times \int \frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} d^3x'$$

$\nabla \bullet \vec{B} = 0$ [the homogeneous equation]

$\nabla \times \vec{B} = \mu_0 \vec{J}$ [the inhomogeneous equation]

Ampère's law

or, $\oint \vec{B} \bullet d\vec{s} = \mu_0 I_{\text{through}}$

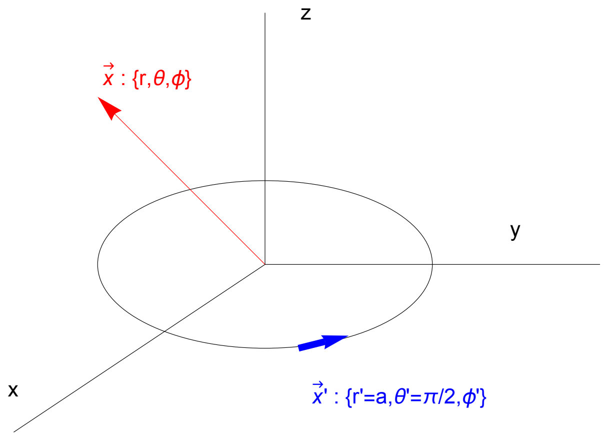
Section 5–4
The Vector Potential

We have div $\vec{B} = 0$.

Therefore we can write $\vec{B} =$ curl $\vec{A}$. Because of gauge invariance, we can also require div $\vec{A} = 0$; this is called *the Coulomb gauge*. Then

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} d^3x'$$

Section 5–5
Example: a circular current loop (Fig 5.5)



$$\vec{J}(\vec{x}) = J_\phi(\theta, \phi)\, \hat{e}_\phi$$

$$J_\phi = \frac{I}{a}\, \delta(\cos\theta)\, \delta(r-a)$$

⟹ the vector potential

$$A_\phi(r,\theta,\phi) = \frac{\mu_0 I}{4\pi a} \int r'^2\, dr'\, d\Omega'$$

$$\cdot \frac{\cos(\phi-\phi')\, \delta(\cos\theta')\, \delta(r'-a)}{\left|\vec{x}-\vec{x}'\right|}$$

$$|\vec{x}-\vec{x}'| = \sqrt{r^2 + (r')^2 - 2\, r\, r' \sin\theta \cos(\phi-\phi')}$$

{the tricky factor $\cos(\phi-\phi') = \hat{e}_\phi(\phi) \bullet \hat{e}_\phi(\phi')$ }
Do the integral . ⟹ Result is

$$\xi = \frac{4\, a r \sin\theta}{a^2+r^2-2\, a r \sin\theta}$$

$$A_\phi(r,\theta) = \frac{\mu_0 I a}{\pi} (a^2+r^2-2\, a\, r \sin\theta)^{-1/2} \cdot$$

$$\cdot \left\{ \frac{2+\xi}{\xi} K(-\xi) - \frac{2}{\xi} E(-\xi) \right\}$$

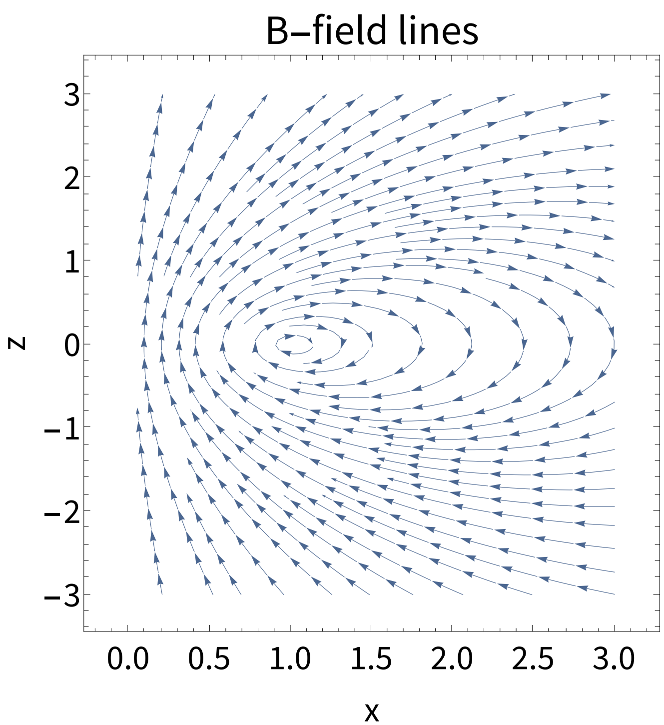
Here K and E are Elliptic Integral Functions, as defined in Mathematica. (I think Jackson uses slightly different notations for these functions.)

Jackson gives some other equations, in terms of spherical harmonic functions, $Y_{\mathrm{lm}}(\theta,\phi)$.
With Mathematica, or some other computer language, the expansion in spherical harmonics is kind of unnecessary.


```
a = 1;
ξ = 4*a*r*Sin[θ]/(a^2 + r^2 - 2*a*r*Sin[θ]);

Aϕ[r_, θ_] = CONST*
  Power[a^2 + r^2 - 2*a*r*Sin[θ], -1/2]*
  ((2 + ξ)/ξ*EllipticK[-ξ] - 2/ξ*EllipticE[-ξ]);

Br[r_, θ_] = 1/(r*Sin[θ])*D[ Sin[θ]*Aϕ[r, θ], θ];
Bθ[r_, θ_] = -1/r*D[ r*Aϕ[r, θ], r];

cc = {r -> Sqrt[x^2 + z^2], θ -> ArcTan[z, x]};
(* check *) {θ /. cc  /. {x -> 1, z -> 0.1}, Tan[1.47113]};

Bx[x_, z_] = Br[r, θ]*Sin[θ] + Bθ[r, θ]*Cos[θ] /. cc;
Bz[x_, z_] = Br[r, θ]*Cos[θ] - Bθ[r, θ]*Sin[θ] /. cc;
(* check *) {Bx[2.0, 0.01], Bz[2.0, 0.01]};
```

```
StreamPlot[{Bx[x, z], Bz[x, z]}, {x, 0.01, 3}, {z, -3, 3},
 FrameLabel -> {"x", "z"}, PlotLabel -> "B-field lines", ImageSize
-> Large, BaseStyle -> ff]
```

Out[•]=



$$\xi = \frac{4\,a r \sin\theta}{a^2+r^2-2\,a r \sin\theta}$$

$$A_\phi(r,\theta) = \frac{\mu_0 I a}{\pi}(a^2+r^2-2\,a\,r\,\sin\theta)^{-1/2}\left\{ \frac{2+\xi}{\xi} K(-\xi) - \frac{2}{\xi} E(-\xi) \right\}$$

The limit a → 0; or, equivanently r ≫ a

$$\xi \approx \frac{4\,a \sin\theta}{r}$$

$$K(-\xi) \approx \frac{\pi}{2} - \frac{\pi\,\xi}{8} + \frac{9\,\pi\,\xi^2}{128}$$

and $E(-\xi) \approx \frac{\pi}{2} + \frac{\pi\,\xi}{8} - \frac{3\,\pi\,\xi^2}{128}$

$$A_\phi(r,\theta) \approx \frac{\mu_0 I a}{\pi r}\left[\frac{2}{\xi}\left(-\frac{\pi\,\xi}{4} + \frac{12\,\pi\,\xi^2}{128}\right) + \frac{\pi}{2} - \frac{\pi\,\xi}{8} \right]$$

$$A_\phi(r,\theta) \approx \frac{\mu_0 I a}{\pi r}\,\frac{\pi\,\xi}{16} = \frac{\mu_0 I \pi a^2}{4\,\pi}\,\frac{\sin\theta}{r^2}$$

which is the vector potential of a pointlike magnetic dipole; $\frac{\mu_0}{4\pi}\,\frac{\vec{m}\times\vec{r}}{r^3}$.