enter

Lecture 2-2 (Mon Sept 16)

Magnetization $\vec{M}$ and the Magnetic Field $\vec{H}$

Jackson: Sections 5.6 5.7 5.8

Section 5.6
Magnetic moment of a localized current distribution

Given $\vec{J}(\vec{x})$,

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x' ;$$

expand in powers of $1/r$ ($r = |\vec{x}|$)

$$= \frac{\mu_0}{4\pi} \{ \frac{1}{r} \int \vec{J}(\vec{x}')\, d^3x' + \frac{1}{r^3} \int \vec{J}(\vec{x}')\, \vec{x} \cdot \vec{x}'\, d^3x' + ...$$

Theorem 1:

$$\int \vec{J}(\vec{x}')\, d^3x' = 0 .$$

Proof: For a localized current,

$$\int \nabla \cdot (x_i \mathbf{J})\, d^3x = \oint \hat{n} \cdot (x_i \vec{J})\, da = 0 ;$$

$$= \int (J_i + x_i \nabla \cdot \vec{J})\, d^3x = \int J_i\, d^3x = 0.$$

The monopole term is zero.

Theorem 2: The next term $O(r^{-2})$ can be simplified, in terms of the magnetic dipole moment $\vec{m}$.

Proof: What is $d_i \equiv \int (\vec{x} \cdot \vec{x}')\, J_i(\vec{x}')\, d^3 x'$?

For a localized current,

$$\int \nabla' \cdot (x'_i\, x'_j\, \vec{J})\, d^3x' = \oint \hat{n} \cdot (x'_i\, x'_j\, \vec{J})\, da' = 0 ;$$

$$= \int (x'_j J_i + x'_i J_j)\, d^3x' = 0$$

Now calculate $[\vec{x} \times (\vec{x}' \times \vec{J})]\, d^3x'$

$$= \int [\vec{x}' (\vec{x} \cdot \vec{J}) - (\vec{x} \cdot \vec{x}') \vec{J}]\, d^3x'$$

$$= -2\, \vec{d}$$

so $\vec{d} = -\frac{1}{2}\, \vec{x} \times \int \vec{x}' \times \vec{J}\, d^3x' = -\vec{x} \times \vec{m}$.

$$\vec{A}^{(2)}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{x}}{r^3}$$

$\vec{m} = \frac{1}{2} \int \vec{x}' \times \vec{J}(\vec{x}')\, d^3x'$	magnetic dipole moment

Section 5.7
Force Torque and Energy

I assume you studied this in PHY 841.

$$\vec{N} = \vec{m} \times \vec{B}$$

$$U = -\vec{m} \bullet \vec{B}$$

Exercise: When is this true (eq 5.57)?

$$|\vec{m}| = I \times \text{area}$$

Section 5.8
The Macroscopic Equations

Note the similarity to dielectrics.

- For electrostatics we defined $\vec{P}(\vec{x})$ and $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$;

  that was because $\rho_{\text{bound}} = -\nabla \bullet \vec{P}$;

  then $\nabla \bullet \vec{D} = \rho_{\text{free}}$.

- Now for magnetostatics we'll define $\vec{M}(\vec{x})$ and $\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$;

  that's because $\vec{J}_{\text{bound}} = \nabla \times \vec{M}$;

  then $\nabla \times \vec{H} = \vec{J}_{\text{free}}$.

  (and of course $\nabla \bullet \vec{B} = 0$)

Derivations

The electric currents inside molecules will be smoothed out by averaging over many molecules in small volumes ΔV.

Recall the vector potential of a magnetic dipole located at $\vec{x}'$ ...

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3} \quad \text{where } \vec{r} = \vec{x} - \vec{x}'$$

Apply this to the net magnetic moment of a volume ΔV.

I.e., replace $\vec{m}$ by $N(\vec{x}')\,\langle\,\vec{m}_{\text{molecule}}\,\rangle\, d^3x'$.

$$\vec{M}(\vec{x}') \equiv N(\vec{x}')\,\langle\,\vec{m}_{\text{molecule}}\,\rangle_{\vec{x}'}$$

= the *magnetization* at $\vec{x}'$

= the magnetic dipole moment density

Free current and bound current

The vector potential due to bound molecular currents in a small volume ΔV located at $\vec{x}'$ is

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{(\vec{m})\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

Thus the vector potential at $\vec{x}$ due to all the currents in the system is

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec{J}_{\text{free}}(\vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec{M}(\vec{x}')\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

where $\vec{M}(\vec{x}') = N(\vec{x}')\left\langle\,\vec{m}\,\right\rangle_{\vec{x}'}$.

Now rewrite the second term

$$= \frac{\mu_0}{4\pi}\int d^3x'\, \vec{M}(\vec{x}')\times\nabla' \frac{1}{|\vec{x}-\vec{x}'|}$$

$$= \frac{\mu_0}{4\pi}\int \nabla' \times \vec{M}(\vec{x}') \frac{d^3x'}{|\vec{x}-\vec{x}'|}$$

+ possible surface term?

$$\therefore \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}_{free}\left(\vec{x}\,'\right) + \nabla' \times \vec{M}\left(\vec{x}\,'\right)}{\left|\vec{x} - \vec{x}\,'\right|} d^3x'$$

I.e., a magnetization $\vec{M}$ that varies with with position makes an effective current density $\vec{J}_M = \nabla\times\vec{M}$.

Bound surface current density

This is important, but it is a little hidden in Jackson. See Equation (5.103).
Start with (5.77), integrate by parts, *and keep the surface integral* ; the result is ...

On any boundary surface S,

$$\vec{K}_M = \vec{M} \times \hat{n}$$

Example: A uniformly magnetized bar of iron is equivalent to a solenoid. (Homework)

The magnetic field $\vec{H}$

Calculate $\vec{B}$ = curl $\vec{A}$; and then curl $\vec{B}$ ...
So, the result is

$$\text{curl } \vec{B} = \mu_0 \vec{J}_{free} + \mu_0 \text{ curl } \vec{M} \, ;$$

and of course we still have $\nabla \bullet \vec{B} = 0$.

The "magnetic field" is defined by

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} .$$

$\Longrightarrow$ From the previous eq,
curl $\vec{H} = (1/\mu_0)(\mu_0 \vec{J}_{free} + \mu_0$ curl $\vec{M})$ − curl $\vec{M} = \vec{J}_{free}$.

$\vec{H}$ is the magnetic field. $\vec{B}$ is often also called the "magnetic field", which could be a source of confusion. More properly, $\vec{B}$ is the "magnetic induction" (Jackson) or the "magnetic flux density" (Faraday).

So, finally, the macroscopic equations for magnetostatics are

$$\text{div}\ \vec{B} = 0 \quad \text{and} \quad \text{curl}\ \vec{H} = \vec{J}_{free}$$

$$\text{where} \quad \text{div}\ \vec{J}_{free} = 0$$

$$\text{and} \quad \vec{B} = \mu_0 (\vec{H} + \vec{M})$$

Constitutive equations

diamagnetic medium	$\vec{B} = \mu \vec{H}$	$\mu < \mu_0$
paramagnetic med.	$\vec{B} = \mu \vec{H}$	$\mu > \mu_0$
ferromagnetic med.	$\vec{B} = \vec{F}(\vec{H})$	nonlinear

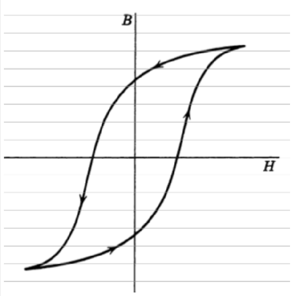

Figure 5.8 Hysteresis loop giving **B** in a ferromagnetic material as a function of **H**.

Boundary Conditions

Jackson Section 5.8

Magnetic induction ; $\vec{B}(\vec{x})$

On any surface in a magnetic system, the normal component of $\vec{B}$ is continuous across the surface. The reason is because div $\vec{B} = 0$.

$$\hat{n} \cdot \vec{B}\,(2) = \hat{n} \cdot \vec{B}\,(1)$$

where $\hat{n}$ = the unit normal vector pointing from region 1 into region 2.

Magnetic field ; $\vec{H}(\vec{x})$

On any surface in a magnetic system, the tangential component of $\vec{H}$ is continuous across the surface if there is no free surface current on the surface. The reason is because curl $\vec{H}$ = 0.

$$\hat{n} \times \vec{H}(2) = \hat{n} \times \vec{H}(1)$$

However, if there is a free surface current $\vec{K}$ on the surface, then

$$\hat{n} \times [\vec{H}(2) - \vec{H}(1)] = \vec{K}$$

(check the units)