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Lecture 2-6 {Wed , Sept 25}
Time-dependent magnetic fields (quasi-static effects)

Faraday's Law of Induction

Jackson Section 5.15

Recall from PHY 841,

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

This is one of the four Maxwell equations, *and it is true in macroscopic media as well as in empty space*.

Faraday was a genius of experimental physics. You should know some of the many ways that he demonstrated that ***a time-dependent magnetic flux creates an electric field***.

Why is there an alternating voltage in every wall socket?

answer: electromagnetic induction

(= in the USA, r.m.s. 120 volts at 60 hertz, as recommended by Nikola Tesla.)

Faraday's law

Written as a partial differential equation, relating $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$,

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

Why is it true?

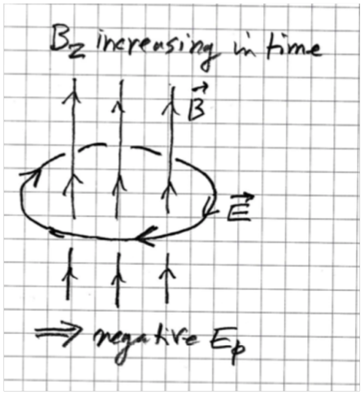
(1) This is a fundamental equation. You can't prove why it is true. There is nothing more fundamental from which to derive it!

(2) Maxwell derived it from the *integral* relation

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \iint \vec{B} \cdot d\vec{a}$$

$$\text{or,} \quad \mathcal{E} = -\frac{d\Phi}{dt}$$

—which describes Faraday's experimental results. (and Lenz's law!)



Proof: Apply Stokes's Theorem;

$$\oint \vec{G} \cdot d\vec{s} = \iint (\nabla \times \vec{G}) \cdot d\vec{a}$$

directions of $d\vec{s}$ and $d\vec{a}$ are related by the right–hand rule

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(3) The homogeneous field equations must be covariant in Minkowski space,

$$\epsilon^{\mu\nu\alpha\beta}\partial_\nu F_{\alpha\beta} = 0$$

Case $\mu = 0$

$\epsilon^{0\,\nu\alpha\beta}\,\partial_\nu\, F_{\alpha\beta} = -\,\epsilon_{ijk}\,\partial_i\, F_{jk}$
$= \partial_i\, \epsilon_{ijk}\, \epsilon_{jkl}\, B^l = 2\,\nabla \cdot \vec{B}$
$\therefore\ \nabla \cdot \vec{B} = 0$

Case $\mu = \mathrm{i}$ (= 1, 2, 3)

$\epsilon^{i\nu\alpha\beta}\,\partial_\nu\, F_{\alpha\beta} = \epsilon_{ijk}\,\partial_0\, F_{jk} + \epsilon_{ijk}\,\partial_j\, F_{0\,k}$
$= \frac{\partial B^i}{c\,\partial t} + \epsilon_{ijk}\,\partial_j\, E^k/c$
$\therefore\ \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

Comments:

- Remember, Faraday's law relates $\vec{E}$ and $\vec{B}$, *not* $\vec{D}$ *or* $\vec{H}$ *!*
- Also, in general $\nabla \times \vec{E} \neq 0$.

Therefore, $\vec{E} \neq -\nabla\Phi$.

- The quasi-static approximation is $\partial\vec{D}/\partial t \approx 0$; i.e., $\ll \vec{J}$.

Energy and the Magnetic Field

Jackson Section 5.16

Recall for the electric field in macroscopic electrostatics,

$$U = \int d^3x \int_0^D \vec{E} \cdot \delta\vec{D}$$

$$= \frac{1}{2} \int \vec{E} \cdot \vec{D} \; d^3x \quad \text{(linear dielectric)}$$

which we derived by assembling a system of point charges.

But for magnetism there are no point charges!

Can we calculate the energy required to assemble a magnetostatic system of current loops?

But there is a problem. *As we bring up another current loop, all the currents will change because of electromagnetic induction*. To keep the current constant, the sources of current must do work.

Theorem

The work that must be supplied to change the vector potential by

$$\vec{A} \to \vec{A} + \delta\vec{A}$$

is $\delta W = \int \delta\vec{A} \cdot \vec{J} \; d^3x$.

Proof : exercise

Or ...

$$\delta W = \int \delta\vec{A} \cdot (\nabla \times \vec{H})\, d^3x$$

$$= \int \vec{H} \cdot (\nabla \times \delta\vec{A})\, d^3x$$

$$= \vec{H} \cdot \delta\vec{B}\, d^3x$$

For a linear medium,
$\vec{H} \bullet \delta\vec{B} = \frac{1}{2}\delta\,(\vec{H}\bullet\vec{B})$, and so

$$W = \frac{1}{2} \int \vec{H} \cdot \vec{B}\, d^3x \quad \text{(linear medium)}$$

See Jackson page 214 for some additional results.

∎1

$$W = \frac{1}{2} \int \vec{J} \cdot \vec{A}\, d^3x$$

∎2 To calculate the *force* on an object in a magnetic field: “It is left as an exercise for the reader...” (energy → force)

∎3 Compare equations (5.150) and (5.72)

$$W = \frac{1}{2} \int_V \vec{M} \cdot \vec{B}_0\, d^3x \qquad (5.150)$$

$$U = -\vec{m} \cdot \vec{B} \qquad (5.72)$$

!!!

anything wrong here?

In fact, issues involving the energy in a magnetic system are sometimes complicated. (homework problems)

$$W = \frac{1}{2} \int \vec{H}.\vec{B}\ d^3 x$$

Empty space: $U = \int \frac{B^2}{2\,\mu_0}\, d^3x$

Material: $U = \int \frac{B^2}{2\,\mu_0}\, d^3x - \int \frac{1}{2}\, \vec{M}.\vec{B}\, d^3x$

We will study this again, in Poynting's theorem.