Topic #3

Jackson Chapter 6

Maxwell Equations,
Macroscopic Electromagnetism,
Conservation Laws



**Lecture 3-1 {Mon , Sept 30}**
**Maxwell's Equations**

Jackson Section 6.1

So far in PHY 842 we have used these field equations, for macroscopic electrostatics, magneto-statics, and quasi-static magnetic fields.

$$\nabla \cdot \vec{B} = 0 \quad \text{and} \quad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\nabla \cdot \vec{D} = \rho \quad \text{where} \quad \vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$$\nabla \times \vec{H} = \vec{J} \quad \text{where} \quad \vec{B} = \mu_0 ( \vec{H} + \vec{M} )$$

*One of these equations is wrong!*

Maxwell derived the field equations (~ 1865) from what was known about electricity and magnetism before his field theory.

And he realized that the equations *are not consistent for general time dependence.*

A mathematical identity :

$\nabla\bullet(\nabla\times\vec{H}) = 0$ ;

so Ampere's Law implies $\nabla\bullet\vec{J} = 0$ ;

but in general $\nabla\bullet\vec{J} = -\frac{\partial\rho}{\partial t} \neq 0$ .



To fix the inconsistency, Maxwell noted that Gauss's law implies

$\frac{\partial\rho}{\partial t} = \nabla\bullet\frac{\partial\vec{D}}{\partial t}$ ;

so we can fix the problem by making this change:

$\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$ .

Verify...

$\nabla\bullet(\nabla\times\vec{H}) = 0$

$= \nabla\bullet\vec{J} + \nabla\bullet\frac{\partial\vec{D}}{\partial t} = \nabla\bullet\vec{J} + \frac{\partial\rho}{\partial t} = 0.$ ✓

Maxwell' s equations

$$\nabla\cdot \vec{B} = 0 \quad \text{and} \quad \nabla\times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\nabla\cdot \vec{D} = \rho \quad \text{where} \quad \vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$$\nabla\times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{where} \quad \vec{B} = \mu_0 ( \vec{H} + \vec{M} )$$

and charge is locally conserved

$$\nabla\cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

$\Longleftrightarrow$ macroscopic classical electrodymamics.



## Vector and Scalar Potentials

Jackson Section 6.2

We know from electrostatics and magnetostatics, we can simplify calculations by introducing *potentials* .

For general time dependence it goes like this:

$$\nabla\cdot \vec{B} = 0 \;\Rightarrow\; \text{write } \vec{B} = \nabla \times \vec{A}$$

$$\nabla\times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 = \nabla\times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

$$\Longrightarrow \text{write } \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla\Phi$$

$$\vec{B} = \nabla \times \vec{A} \quad \text{and} \quad \vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}$$

## Gauge Transformations

Jackson Section 6.3

$\vec{A}(\vec{x},t)$ and $\Phi(\vec{x},t)$ are not unique.

Consider these *transformations* of $\vec{A}$ and $\Phi$,

$$\vec{A} \longrightarrow \vec{A}' = \vec{A} + \nabla\lambda$$

$$\Phi \longrightarrow \Phi' = \Phi - \frac{\partial\lambda}{\partial t}$$

where $\lambda(\vec{x},t)$ is ***any*** *scalar function*

**Theorem**. These new potentials $\vec{A}'$ and $\Phi'$ describe the same $\vec{E}$ and $\vec{B}$ fields as $\vec{A}$ and $\Phi$.



**Proof**

$$\vec{B}' = \nabla\times\vec{A}' = \nabla\times\vec{A} = \vec{B}$$

because $\nabla\times\nabla\lambda = 0$

$$\vec{E}' = -\nabla\Phi' - \frac{\partial\vec{A}'}{\partial t} = -\nabla\Phi - \frac{\partial\vec{A}}{\partial t} = \vec{E}$$

because $-\nabla\left(-\frac{\partial\lambda}{\partial t}\right) - \frac{\partial}{\partial t}\left(\nabla\lambda\right) = 0$

Because the potentials are not unique, we can impose another condition on them — which is called “a gauge condition”.
In other words, if necessary apply a gauge transformation — *ie*, choose $\lambda(\vec{x},t)$ — such that the gauge condition is satisfied.

## The Lorenz (or, Lorentz) gauge:

$$\nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0$$

Then Maxwell's equations imply

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -\rho/\epsilon_0$$

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

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In empty space,

$$\vec{B} = \nabla \times \vec{A} \quad \text{and} \quad \vec{E} := -\nabla \Phi - \frac{\partial \vec{A}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \left( \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

$\vec{B} = \mu_0 \vec{H}$ $\qquad$ $\vec{D} = \epsilon_0 \vec{E}$

$$LHS = \nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$RHS = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial}{\partial t}\left(-\nabla \Phi - \frac{\partial \vec{A}}{\partial t}\right)$$

$\mu_0 \epsilon_0 = 1/c^2$

$$= -\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} + \mu_0 \vec{J} - \frac{1}{c^2}\nabla\left(\frac{\partial \Phi}{\partial t}\right)$$

$$-\nabla^2 \vec{A} + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \mu_0 \vec{J} - \nabla \underbrace{\left(\frac{1}{c^2}\frac{\partial \Phi}{\partial t} - \nabla \cdot \vec{A}\right)}_{=0 \text{ in Lorenz gauge}}$$



## The Coulomb (or, transverse) gauge:

$$\nabla \cdot \vec{A} = 0$$

Then Maxwell's equations imply

$$\nabla^2 \Phi = -\rho/\epsilon_0$$

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}_{\text{transverse}}$$

The Coulomb gauge is sometimes called the “radiation gauge”.

## The Lorenz (or, Lorentz) gauge:

$$\nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0$$

Then Maxwell's equations imply

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\rho/\epsilon_0$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\,\vec{J}$$

Advantages and disadvantages:
(A) Causality is manifestly true
because all potential components propagate with velocity c;
(A) Lorentz invariance is manifestly true
because $A^\mu = \{\Phi, \vec{A}\}$ is a 4 vector.
(D) there are unphysical wave modes (timelike and longitudinal).



## The Coulomb (or, transverse) gauge:

$$\nabla\cdot\vec{A} = 0$$

Then Maxwell's equations imply

$$\nabla^2\Phi = -\rho/\epsilon_0$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\,\vec{J}_{\text{transverse}}$$

Advantages and disadvantages:
(A) does not introduce unphysical wave modes;
(D) the gauge condition violates Lorentz invariance;
(D) causality is not manifestly true because
$\nabla^2\Phi(\vec{x},t) = -\rho(\vec{x},t)$ violates causality.

The disadvantages create significant problems in quantum field theories, like Q.E.D. and Q.C.D.

## Gauge invariance in high-energy physics

The three fundamental parts of the Standard Model:

- Q.E.D. is a U(1) gauge theory.
- The electro-weak unified theory is an $SU(2) \times U(1)$ gauge theory, with spontaneous symmetry breaking.
- Q.C.D. is an SU(3) gauge theory with respect to color transformations.

No one knows why all the fundamental interactions (except gravity) are gauge theories.