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Lecture 3-2 { Wed , Oct 2 }

Derivation of the equations of macroscopic electromagnetism

Jackson Section 6.6

$$\nabla \cdot \vec{B} = 0 \quad \text{and} \quad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\nabla \cdot \vec{D} = \rho \quad \text{and} \quad \nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}$$

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \quad \text{and} \quad \vec{B} = \mu_0 (\vec{H} + \vec{M})$$

Remember, ρ and $\vec{J}$ here are the macroscopic (i.e., ***free***) charge and current densities.

Where did these equations come from?

From the mind of Maxwell (~1865)

What did he use?

- Gauss & Ampere
- Faraday's "lines of force" became Maxwell's idea of the ***fields***.
- Atoms and molecules carry internal charges.
- The ***aether ;*** he thought that the fields are stresses and strains in the aether.

But some things Maxwell did not know—

- the electron (Thomson, 1897)
- atomic structure with nuclei (Rutherford, 1909)
- There is no aether (Einstein, 1905)

In Section 6.6, Jackson provides a *more rigorous derivation* of the macroscopic equations — more rigorous than the previous derivations.

The microscopic world

(1) Write the field equations for microscopic fields $\vec{e}$ and $\vec{b}$, and microscopic sources η and $\vec{j}$ ...

$$\nabla \cdot \vec{b} = 0 \quad \text{and} \quad \nabla \times \vec{e} + \frac{\partial \vec{b}}{\partial t} = 0$$

$$\nabla \cdot \vec{e} = \eta / \epsilon_0 \quad \text{and} \quad \nabla \times \vec{b} - \frac{1}{c^2}\frac{\partial \vec{e}}{\partial t} = \mu_0 \vec{j}$$

(2) And now average over small but macroscopic regions of space.

How large is "large"?

Jackson's Estimate

$$L_0 = 10^{-8}\,\text{m} = 10^2\,\mathring{A}$$

= lower limit of a macroscopic length;

time averaging is not necessary.

How to do the averaging

Given a function $F(\vec{x}, t)$ with singularities due to atomic dimensions, define the spatial average of F with respect to a test function $f(\vec{x})$ by

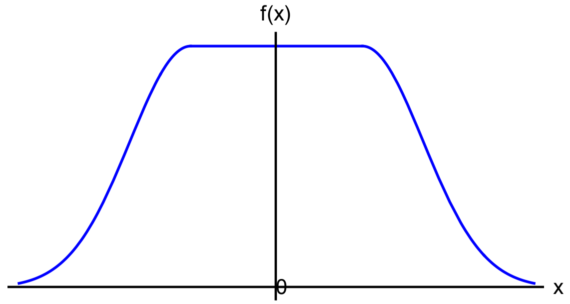
$$\langle F(\vec{x},t)\rangle = \int d^3x' \, f(\vec{x}') \, F(\vec{x} - \vec{x}')$$

$$\int d^3x' \, f(\vec{x}') = 1$$

We want $f(\vec{x})$ to smooth out all the short-range fluctuations of $F(\vec{x},t)$. "Short range" means distances $< L_0$.

So $f(\vec{x})$ should focus on a length scale $L > L_0$.

Schematic diagram of a test function $f(\vec{x})$ used for the spatial averaging; $L \gg a$ and $\Delta L \gg a$.



Examples of $f(\vec{x})$ could be:

- Example 1 - average over a sphere of radius R

$$f(\vec{x}) = \frac{3}{4\pi R^3}\,\Theta(R-r)$$

$\Theta(\zeta)$ = Heaviside Theta Function

However, the sharp cut-off at r = R might produce short-range "jitter".

- Example 2 - Gaussian averaging

$$f(\vec{x}) = (\pi R^2)^{-3/2}\, e^{-r^2/R^2}$$

is better—a smooth test function.

In both of these examples R is small but macroscopic, so the region of the integral contains many molecules.

Because molecule are tiny, the precise form of the test function does not matter.

Apply the averaging procedure to Maxwell's equations

First, note this identity,

$$\frac{\partial}{\partial x_i}\langle F(\vec{x},t)\rangle = \int d^3x'\, f(\vec{x}')\, \frac{\partial F}{\partial x_i}(\vec{x}-\vec{x}',t)$$

$$= \left\langle \frac{\partial F}{\partial x_i} \right\rangle$$

The macroscopic fields are defined by

$$\vec{E}(\vec{x},t) = \langle \vec{e}(\vec{x},t) \rangle$$

$$\vec{B}(\vec{x},t) = \langle \vec{b}(\vec{x},t) \rangle$$

The homogeneous Maxwell equations are easy,

$$\langle \nabla\cdot \vec{b}\rangle = 0 \Longrightarrow \nabla\cdot \vec{B} = 0$$

$$\left\langle \nabla\times \vec{e} + \frac{\partial \vec{b}}{\partial t} \right\rangle = 0 \Longrightarrow \nabla\times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

The inhomogeneous Maxwell equations

$$\epsilon_0 \nabla \cdot \vec{E} = \langle \eta(\vec{x},t) \rangle$$

$$\frac{1}{\mu_0} \nabla \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \langle \vec{j}(\vec{x},t) \rangle$$

So now we need to calculate

$\langle \eta \rangle$ and $\langle \vec{j} \rangle$.

The medium may have *free charge* (not belonging to the molecules that make up the medium) and *bound charge* ≡ the charge that belongs to the molecules.

Charge density

By definition

$$\eta(\vec{x},t) = \sum_i q_i \, \delta^3(\vec{x} - \vec{x}_i)$$

$$\eta_{free}(\vec{x},t) = \sum_{j\,(free)} q_j \, \delta^3(\vec{x} - \vec{x}_j)$$

$$\eta_{bound}(\vec{x},t) = \sum_{n\,(mol)} \eta_n(\vec{x},t)$$

where n is the label for the n^{th} molecule; and

$$\eta_n(\vec{x},t) = \sum_{i \in n} q_i \, \delta^3(\vec{x} - \vec{x}_i)$$

Now, apply test-function averaging to a single molecule,

$$\langle \eta(\vec{x}, t) \rangle = \int d^3x' \, f(\vec{x}') \, \eta_n(\vec{x} - \vec{x}', t)$$

$$= \sum_{i \in n} q_i \int d^3x' \, f(\vec{x}') \, \delta^3(\vec{x} - \vec{x}' - \vec{x}_n - \vec{x}_{ni})$$

$$= \sum_{i \in n} q_i \, f(\vec{x} - \vec{x}_n - \vec{x}_{ni})$$

$\Rightarrow$ multipole expansion

$$= \sum_{i \in n} q_i \, f(\vec{x} - x_n) - \vec{p}_n \cdot \nabla f(\vec{x} - \vec{x}_n)$$

$$+ \frac{1}{6} \sum_{\alpha\beta} (Q_n')_{\alpha\beta} \frac{\partial^2 f\left(\vec{x} - \vec{x}_n\right)}{\partial x_\alpha \, \partial x_\beta} + \ldots$$

ie, monopole + dipole + quadrupole + ...

So the averaged charge density is expressed as a sum of multipoles for macroscopic phenomena.

So far, this is for one molecule. Now sum the molecules in the material $\Longrightarrow$

Results

The averaged microscopic charge density is

$$\langle \eta(\vec{x},t) \rangle = \rho(\vec{x},t) - \nabla \cdot \vec{P}(\vec{x},t) + \nabla\nabla \cdot\cdot \overleftrightarrow{Q}'(\vec{x},t) + \ldots$$

$$\rho(\vec{x},t) = \langle \sum_{j\,(\text{free})} q_j \, \delta^3(\vec{x} - \vec{x}_j) + \sum_{n\,(\text{mol})} q_n \, \delta^3(\vec{x} - \vec{x}_n) \rangle$$

$$\vec{P}(\vec{x},t) = \langle \sum_{n\,(\text{mol})} \vec{p}_n \, \delta^3(\vec{x} - \vec{x}_n) \rangle$$

$$\overleftrightarrow{Q}'(\vec{x},t) = \frac{1}{6} \langle \sum_{n\,(\text{mol})} \overleftrightarrow{Q}_n' \, \delta^3(\vec{x} - \vec{x}_n) \rangle$$

Displacement Field

$$\epsilon_0 \nabla \cdot \vec{E} = \langle \eta(\vec{x},t) \rangle$$

We want to define the displacement field such that $\nabla \cdot \vec{D} = \rho$. Therefore,

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} - \sum_\beta \frac{\partial Q'_{\alpha\beta}}{\partial x_\beta} \hat{e}_\beta + \ldots$$

Normally, $\vec{D} \approx \epsilon_0 \vec{E} + \vec{P}$ is good enough.

$\langle \vec{j}(\vec{x},t) \rangle$

By similar calculations,
"leaving the gory details to a problem for readers who enjoy such challenges" analyze the microscopic current density,

$$\vec{j}(\vec{x},t) = \sum_i q_i \vec{v}_i \delta^3(\vec{x} - x_j(t))$$

$$\vec{v}_i = d\vec{x}_i/dt$$

The final result of the derivation is

$$\frac{1}{\mu_0}\vec{B} - \vec{H} = \vec{M} + (\vec{D} - \epsilon_0 \vec{E}) \times \vec{v}$$

where $\vec{v}$ is the velocity of a medium in motion. Normally,

$$\frac{1}{\mu_0}\vec{B} - \vec{H} = \vec{M}\,.$$

Jackson finishes Section 6.6 (pages 257-258) with some additional comments, mainly for the experts.

Homework assignment 5 is due Friday.