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Lecture 3-3 { Fri , Oct 4 }

Poynting's Theorem

Jackson Sections 6.7, 6.8, (6.9)

$$\nabla\cdot\vec{B} = 0 \quad \text{and} \quad \nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

$$\nabla\cdot\vec{D} = \rho \quad \text{and} \quad \nabla\times\vec{H} - \frac{\partial\vec{D}}{\partial t} = \vec{J}$$

$$\vec{D} = \epsilon_0\,\vec{E} + \vec{P} \quad \text{and} \quad \vec{B} = \mu_0\,(\vec{H} + \vec{M})$$

Here ρ and $\vec{J}$ are the macroscopic (i.e., ***free***) charge and current densities.

Theorem 1

$\vec{E}\cdot\vec{J}$ is the work *per unit time per unit volume,* done on the macroscopic charge by the electric field.

That is, $\vec{E}\cdot\vec{J}$ is the rate of energy conversion from the fields to the free charges, per unit volume.

Proof.

Consider a single charge.

$$\vec{F} = q\,\vec{E} + q\,\vec{v}\times\vec{B}$$

$$\delta W = \int \vec{F}\cdot d\vec{s} = \vec{F}\cdot\vec{v}\,\delta t$$

$$\delta W = q\,\vec{E}\cdot\vec{v}\,\delta t$$

for a single charge

Now, for all the charge in a volume δV,

$$\frac{\delta W}{\delta t} = \sum_i q\, \vec{E}(\vec{x}_i) \bullet \vec{v}_i$$

$$= \int d^3x\, \vec{E}(\vec{x}) \bullet q\, \vec{v}_i\, \delta^3(\vec{x} - \vec{x}_i)$$

$$= \delta V\, \vec{E}(x) \bullet \vec{J}(x)$$

check the units

$$\frac{\delta W / \delta t}{\delta V} = \vec{E} \bullet \vec{J}\,, \quad \text{as claimed.}$$

Conservation of Energy for an"ideal" linear material

Let $u(\vec{x},t)$ be the energy density,

and let $\vec{S}(\vec{x},t)$ be the energy flux.

Units: $[u] = E / L^3$ and $[S] = E / L^2/T$.

Energy is conserved.

(That statement is really the *definition* of energy.)

In a field theory, energy is *locally* conserved.

Therefore,

$$\frac{\partial u}{\partial t} = -\vec{E} \bullet \vec{J} - \nabla \bullet \vec{S}$$

$$\frac{\partial u}{\partial t} = -\vec{E} \cdot \vec{J} - \nabla \cdot \vec{S}$$

Derivation and comments

■ This equation resembles the continuity equation for charge,

$\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{J}$.

However, charge cannot change, whereas energy does change when work is done.

■ Consider a small volume δV, and a small time interval δt. Let U be the electrodynamic energy in δV. Then during time δt,

$$\delta U = \left(\frac{\partial u}{\partial t}\right) \delta t\, \delta V$$
$$- \vec{E} \cdot \vec{J}\, \delta t\, \delta V - \oint \vec{S} \cdot \hat{n}\, da\, \delta t$$

i.e., δU = the increase of electromagnetic energy, minus the work done on the free charge, minus the amount of

energy that flowed out of the surface; the third term is

$$-\oint \vec{S} \cdot \hat{n}\, da = -\int d^3x\, \nabla \cdot \vec{S}$$

by Gauss's theorem

$$= -\nabla \cdot \vec{S}\ \delta V$$

Thus,

$$\frac{\partial u}{\partial t} = -\vec{E} \cdot \vec{J} - \nabla \cdot \vec{S}$$

$\Leftrightarrow$ local conservation of energy

Poynting's Theorem

Here we'll consider a macroscopic medium with $\vec{D}(\vec{x},t) = \epsilon\, \vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t) = \mu\, \vec{H}(\vec{x},t)$, where ϵ and μ are real and constant. *(This is not always true!)*

Poynting's theorem

$$u = \frac{1}{2}\, \vec{E} \bullet \vec{D} + \frac{1}{2}\, \vec{B} \bullet \vec{H}$$

$$\vec{S} = \vec{E} \times \vec{H}$$

Proof

Start with $u = \frac{1}{2}\, \vec{E} \bullet \vec{D} + \frac{1}{2}\, \vec{B} \bullet \vec{H}$ and calculate $\partial u/\partial t$.

(proof *a posteriori*)

$$\frac{\partial u}{\partial t} = \frac{1}{2}\left(\frac{\partial \vec{E}}{\partial t} \bullet \vec{D} + \vec{E} \bullet \frac{\partial \vec{D}}{\partial t}\right) + \frac{1}{2}\left(\frac{\partial \vec{B}}{\partial t} \bullet \vec{H} + \vec{B} \bullet \frac{\partial \vec{H}}{\partial t}\right)$$

$$= \vec{E} \bullet \frac{\partial \vec{D}}{\partial t} + \vec{H} \bullet \frac{\partial \vec{B}}{\partial t}$$

for linear materials

$$\frac{\partial u}{\partial t} = \vec{E} \bullet (\nabla \times \vec{H} - \vec{J}) + \vec{H} \bullet (-\nabla \times \vec{E})$$

$$= -\vec{E} \bullet \vec{J} + \vec{E} \bullet (\nabla \times \vec{H}) - \vec{H} \bullet (\nabla \times \vec{E})$$

$$= -\vec{E} \bullet \vec{J} - \nabla \bullet (\vec{E} \times \vec{H})$$

vector calculus identity

So this is just a property of Maxwell's equations, for linear materials,

$$\frac{\partial u}{\partial t} = -\vec{E} \cdot \vec{J} - \nabla \cdot \vec{S}$$

where $u = \frac{1}{2}\vec{E} \cdot \vec{D} + \frac{1}{2}\vec{B} \cdot \vec{H}$

and $\vec{S} = \vec{E} \times \vec{H}$.

Momentum density

Derivation for systems in "empty space"

$$\frac{d\vec{P}_{mech}}{dt} = \int_V d^3x\,(\rho\vec{E} + \vec{J} \times \vec{B})$$

substitutions from Maxwell's equations ⇒

$$\frac{d\vec{P}_{mech}}{dt} + \frac{d\vec{P}_{field}}{dt} = \oint_S \overleftrightarrow{T} \cdot \hat{n}\, da \quad [1]$$

$$\vec{P}_{field} = \mu_0 \epsilon_0 \int_V \vec{E} \times \vec{H}\; d^3x \quad [2]$$

$$\overleftrightarrow{T} = = \epsilon_0 \{\vec{E}\vec{E} + c^2 \vec{B}\vec{B} - \tfrac{1}{2}(E^2 + c^2 B^2)\overleftrightarrow{1}\} \quad [3]$$

[1] The equation for conservation of momentum

[2] Momentum density = $\vec{g} = \frac{1}{c^2}\vec{E} \times \vec{H}$

$$\vec{g} = \frac{1}{c^2}\vec{S}$$

[3] $\overleftrightarrow{T}$ = the Maxwell stress tensor

Section 6.8
Poynting's theorem in Linear Dispersive Media with Losses

- Ideal materials may have $\vec{D} = \epsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$, where ϵ and μ are real constants.

Real materials are not so simple.

- First, we must separate frequencies; Fourier analysis

$$\vec{E}(\vec{x},t) = \int_{-\infty}^{\infty} d\omega\, \vec{E}(\vec{x},\omega)\, e^{-i\omega t}$$

$$\vec{D}(\vec{x},t) = \int_{-\infty}^{\infty} d\omega\, \vec{D}(\vec{x},\omega)\, e^{-i\omega t}$$

and assume linearity

$$\vec{D}(\vec{x},\omega) = \epsilon(\omega)\, \vec{E}(\vec{x},\omega)$$

sim. $\vec{B}(\vec{x},\omega)= \mu(\omega)\, \vec{H}(\vec{x},\omega\}$

- Reality constraints

$$\vec{E}(\vec{x},-\omega) = \vec{E}^*(\vec{x},\omega)$$

$$\vec{D}(\vec{x},-\omega) = \vec{D}^*(\vec{x},\omega)$$

$$\epsilon(-\omega)= \epsilon^*(\omega)$$

- Now $\vec{E}\bullet(\partial \vec{D}/\partial t) \neq \frac{1}{2}\partial(\vec{E}\bullet \vec{D})/\partial t$.

Calculate $\vec{E}\bullet\frac{d\vec{D}}{\partial t}$

$$= \int d\omega \int d\omega'\, \vec{E}^*(\omega')\, [-i\omega\epsilon(\omega)]\bullet \vec{E}(\omega)\, e^{-i(\omega-\omega')t}$$

- Now make an assumption—that the important range of frequencies is peaked at $\omega = \omega_0$ ...

- After a bit of analysis, we get a new Poynting's theorem,

$$\frac{\partial u_{eff}}{\partial t} + \nabla \cdot \vec{S} + \vec{J} \cdot \vec{E}$$
$$= -\,\omega_0\,\mathrm{Im}[\epsilon(\omega_0)]\,(E_{RMS})^2$$
$$-\,\omega_0\,\mathrm{Im}[\mu(\omega_0)]\,(H_{RMS})^2$$

"Energy" is not conserved if $\epsilon(\omega)$ or $\mu(\omega)$ has a nonzero imaginary part.

- "absorption of energy" or "absorptive dissipation"
- We'll see how atoms of molecules absorb energy in a later lecture.