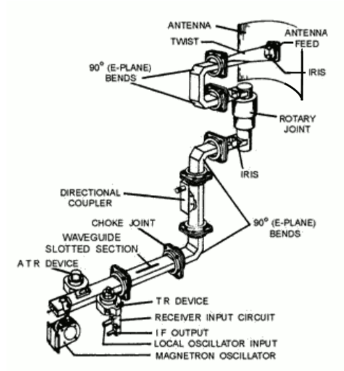
ente

TE modes in a rectangular wave guide

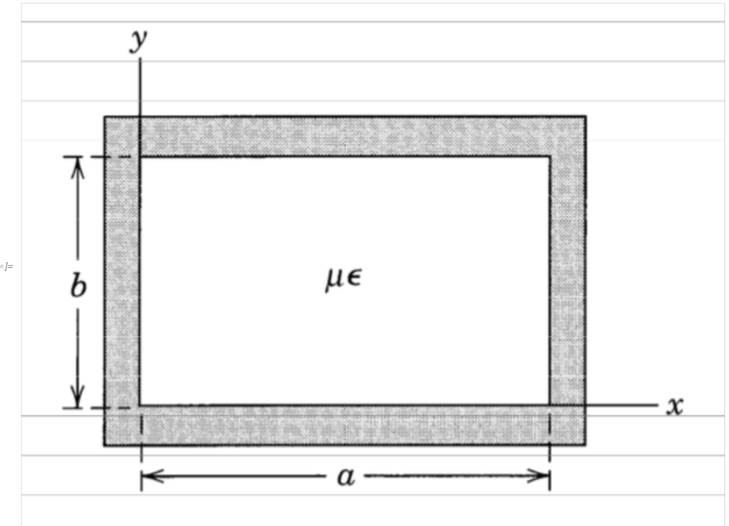
Out[]=



The waveguide is a hollow cylinder with a rectangular cross sectional area

In[]:= f85

Out[]=



We seek to solve Maxwell's equations for fields with these forms

Out[∘]=

$$\vec{E}(\vec{x},t) = \{\, \hat{e}_x\, E_x(x,y) + \hat{e}_y\, E_y(x,y)\,\}\, e^{i\,(kz-\omega t)}$$

$$\vec{B}(\vec{x},t) = \{\, \hat{e}_x\, Bx(x,y) + \hat{e}_y\, B_y(x,y)$$
$$+\, \hat{e}_z\, \psi(x,y)\,\}\, e^{i\,(kz-\omega t)}$$

Comments:

- Wave propagation in the z direction
- $E_z = 0$ (TE polarization)
- $B_z = \psi$(x,y) is the function we are seeking
- In Maxwell's equations,

$$\frac{\partial}{\partial z} \to \text{ik} \quad \text{and} \quad \frac{\partial}{\partial t} = -\text{i}\omega$$

- Define $\nabla_T = \hat{e}_x\, \frac{\partial}{\partial x} + \hat{e}_y\, \frac{\partial}{\partial y}$;

and transverse vectors like $\vec{E}_T = \hat{e}_x\, E_x + \hat{e}_y\, E_y$

Maxwell's Equations

Out[∘]=

$\nabla\bullet\, \epsilon\vec{E} = 0$	$\nabla_T\bullet\vec{E}_T = 0$
$\nabla\bullet\, \mu\vec{H} = 0$	$\nabla_T\bullet\vec{H}_T + ik\,\psi = 0$
$-\frac{\partial(\mu\,\vec{H})}{\partial t} = \nabla\times\vec{E}$	$i\omega\mu\,\psi = \hat{e}_z\bullet(\nabla_T\times\vec{E}_T)$
\`	$i\omega\mu\,\vec{H}_T = ik\,\hat{e}_z\times\vec{E}_T$
$\frac{\partial(\epsilon\,\vec{E})}{\partial t} = \nabla\times\vec{H}$	$0 = \hat{e}_z\bullet(\nabla_T\times\vec{H}_T)$
\`	$-i\omega\epsilon\,\vec{E}_T = -\hat{e}_z\times\nabla_T\psi + ik\,\hat{e}_z\times\vec{H}_T$

The function ψ(x,y) determines everything.

$E_z = 0$ (TE mode)

$H_z = \psi$ (definition of ψ)

- $\boldsymbol{H_T}$

$$i\omega\mu\,\vec{H}_T = ik\,\hat{e}_z \times \vec{E}_T$$

Out[-]=

$$-i\omega\epsilon\,\vec{E}_T = -\hat{e}_z \times \nabla_T\psi + ik\,\hat{e}_z \times \vec{H}_T$$

imply $\vec{H}_T = \frac{ik}{\omega^2\,\mu\epsilon - k^2}\,\nabla_T\psi \equiv \frac{ik}{\gamma^2}\,\nabla_T\psi$

where $\gamma^2 \equiv \mu\epsilon\,\omega^2 - k^2$.

- $\boldsymbol{E_T}$

$$i\omega\mu\,\vec{H}_T = ik\,\hat{e}_z \times \vec{E}_T$$

Out[-]=

implies $\vec{E}_T = \frac{i\omega\mu}{\gamma^2}\,\hat{e}_z \times \nabla_T\psi$

$E_z = 0$ (TE mode)

$H_z = \psi$ (definition of ψ)

$$\vec{H}_T = \frac{ik}{\gamma^2}\,\nabla_T\,\psi$$

$$\vec{E}_T = \frac{-i\omega\mu}{\gamma^2}\,\hat{e}_z \times \nabla_T\,\psi$$

Everything is determined by ψ(x,y).

Solve for ψ(x,y)

The Helmholtz equation

$$(\nabla_T^2 + \gamma^2)\,\psi = 0$$

Proof:

We have $\nabla_T \cdot \vec{H}_T + ik\, H_z = 0$

Out[ ]=

$$= \nabla_T \cdot \left(\frac{ik}{\gamma^2}\nabla_T \psi\right) + ik\,\psi$$

$$= ik\,\frac{\nabla_T^2 \psi + \gamma^2 \psi}{\gamma^2}$$

$$\therefore (\nabla_T^2 + \gamma^2)\,\psi = 0$$

We also need a *boundary condition*.
For TE modes, the Neumann boundary condition, $\partial\psi/\partial n = 0$ on the surface of the conductor. *(Why?)*

The rectangular waveguide

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \gamma^2\psi = 0$$

- separtion of variables, ψ(x,y)=M(x) N(y)

Out[ ]=

$$\frac{M''(x)}{M(x)} + \frac{N''(y)}{N(y)} = -\gamma^2$$

$$\frac{M''(x)}{M(x)} = -\kappa_a^2 \text{ and } \frac{N''(y)}{N(y)} = -\kappa_b^2$$

$$\text{where } \kappa_a^2 + \kappa_b^2 = \gamma^2 = \mu\epsilon\omega^2 - k^2$$

Out[ ]=

$$M(x) = \cos(\kappa_a x) \text{ where } \kappa_a = \frac{m\pi}{a}$$

$$N(y) = \cos(\kappa_b y) \text{ where } \kappa_b = \frac{n\pi}{b}$$

where m and n are integers (not both 0)

Note: cos(κ_ax) obeys the Neumann boundary condition at x = 0 and a. Similarly, cos(κ_by) .

Result

Each TE mode is characterized by two integers {m,n}; and

$$\psi_{mn}(x,y) = H_0 \cos[\frac{m\pi x}{a}] \cos[\frac{n\pi y}{b}]$$

Out[ ]=

$$\gamma_{mn}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

Cutoff frequencies; $\gamma^2 = \mu\epsilon\omega^2 - k^2$

Out[ ]=

$$k^2 = \mu\epsilon\, \omega^2 - \gamma_{mn}^2$$

A propagating wave requires $k^2 > 0$. Otherwise k = i$\alpha \Longrightarrow e^{i(kz-\omega t)} = e^{-\alpha z}\, e^{-i\omega\, t}$ ⇒ the field decreases exponentially as z increases.

Out[ ]=

$$\mu\epsilon\, \omega^2 = \gamma_{mn}^2 + k^2 \text{ must be } > \gamma_{mn}^2$$

Propagation in the waveguide requires $\omega > \omega_{mn} \equiv$ the cutoff frequency

Out[ ]=

$$\omega_{mn} = \frac{\gamma_{mn}}{\sqrt{\mu\epsilon}}$$

The dominant mode

Assume a > b.
The TE mode with the lowest cutoff frequency (≡ *the dominant mode*) has {m,n} = {1,0}.

The fields for the TE_{10} mode are

Out[ ]=

$$H_z^{(10)} = H_0 \cos(\pi x/a)\, e^{i(kz-\omega t)}$$

$$H_x^{(10)} = -\frac{ika}{\pi} H_0 \sin(\pi x/a)\, e^{i(kz-\omega t)}$$

$$E_y^{(10)} = \frac{i\omega a \mu}{\pi} H_0 \sin(\pi x/a)\, e^{i(kz-\omega t)}$$

where $k = \sqrt{\mu\epsilon}\,\sqrt{\omega^2 - \omega_{10}^2}$ and $\omega_{10} = \frac{\pi/a}{\sqrt{\mu\epsilon}}$.

Homework

Jackson's example : Suppose a = 2 b

In[ ]:=

```
{aa, bb} = {1.0, 0.5};
Do[Do[
  ωc[m, n] = Sqrt[Pi^2 * (m^2 / aa^2 + n^2 / bb^2)], {m, 0, 6}], {n, 0, 3}]
Ωc = Table[ SetPrecision[ ωc[m, n] / ωc[1, 0], 4] , {m, 0, 6}, {n, 0, 3}];
Ωc // TableForm
```

Out[ ]//TableForm=

0	2.000	4.000	6.000
1.000	2.236	4.123	6.083
2.000	2.828	4.472	6.325
3.000	3.606	5.000	6.708
4.000	4.472	5.657	7.211
5.000	5.385	6.403	7.810
6.000	6.325	7.211	8.485

Cutoff frequencies ω_{mn}/ω_{10} for a rectangular waveguide with a = 2b .

In[ ]:= jtbl

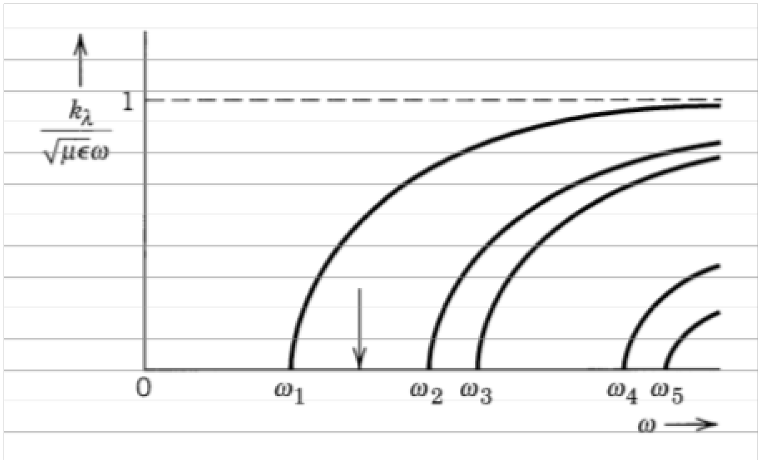
Out[ ]=

m \ *n*	0	1	2	3
0		2.00	4.00	6.00
1	1.00	2.24	4.13	
2	2.00	2.84	4.48	
3	3.00	3.61	5.00	
4	4.00	4.48	5.66	
5	5.00	5.39		
6	6.00			

Figure 8.4

In[•]:= f84
f84b

Out[•]=



Out[•]=

Figure 8.4 Wave number k_λ versus frequency ω for various modes λ. ω_λ is the cutoff frequency.

11-1. In physics, what is ether? And what is ethernet?

11-2. What are the frequencies used by your cell phone?

11-3. What are the frequencies used for WiFi communication?

11-4. Jackson Problem 8.2

11-5. What is the impedance of free space?

The next problems concern a rectangular waveguide with $\delta x = a = 5$ cm and $\delta y = b = 2.5$ cm; also, $\mu = \mu_0$ and $\epsilon = \epsilon_0$.

11-6. Calculate the cutoff frequency and the corresponding wavelength.
11-7. Calculate the energy flux of the TE_{10} mode.
[Hint: the fields are the Real Parts of the complex functions in (8.46).
Take the real parts before you calculate the Poynting vector.]
11-8. Calculate the cutoff frequencies for the TM modes. Hand in a Table like the table below (8.46);
the elements of the table should be $\omega(TM_{mn}) / \omega(TE_{10})$.
11-9. For the waveguide mode TE_{32} ...
hand in a sketch (better: a computer graphic) of the effective surface current density $\boldsymbol{K}$(x,y)
at the wall of the waveguide at y = 0, for $\omega = 2\,\omega_{32}$.