ente

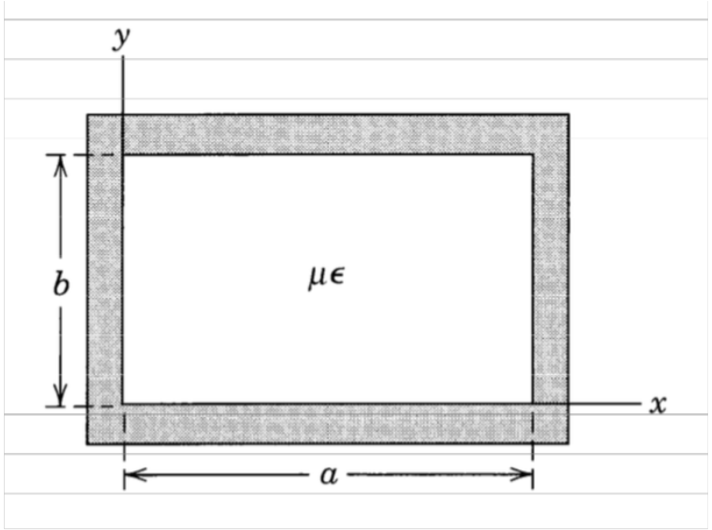
Energy flow and attenuation in a waveguide Section 8.5

The waveguide is a hollow cylinder, centered around the z axis, with an arbitrary cross section.

Rectangular example ...

f85

Out[•]=



We seek to solve Maxwell’s equations for fields with these forms

Out[•]=

$$\vec{E}(\vec{x},t) = \{ \vec{E}_T(x,y) + \hat{e}_z\, \psi_E(x,y) \}\, e^{i\,(kz-\omega t)}$$

$$\vec{B}(\vec{x},t) = \{ \vec{B}_T(x,y) + \hat{e}_z\, \psi_B(x,y) \}\, e^{i\,(kz-\omega t)}$$

where $\psi_E = 0$ (TE mode) or $\psi_B = 0$ (TM mode).

$\vec{E}_T = \hat{e}_x\, E_x + \hat{e}_y\, E_y$, etc.

Comments:

- Wave propagation in the z direction
- $E_z = 0$ (TE polarization) or $B_z = 0$ (TM polarization)
- $B_z = \psi(x,y)$ or $E_z = \psi(x,y)$
- In Maxwell's equations,

$$\frac{\partial}{\partial z} \to ik \quad \text{and} \quad \frac{\partial}{\partial t} = -i\omega$$

- Define $\nabla_T = \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y}$;

and transverse vectors like $\vec{E}_T = \hat{e}_x\, E_x + \hat{e}_y\, E_y$

Maxwell's Equations

Out[∘]=

	TE polarization ($E_z = 0$, $B_z = \psi$)
$\nabla\cdot\, \epsilon\vec{E} = 0$	$\nabla_T\cdot\vec{E}_T = 0$
$\nabla\cdot\, \mu\vec{H} = 0$	$\nabla_T\cdot\vec{H}_T + ik\,\psi = 0$
$-\frac{\partial\left(\mu\,\vec{H}\right)}{\partial t} = \nabla\times\vec{E}$	$i\omega\mu\,\psi = \hat{e}_z\cdot(\nabla_T\times\vec{E}_T)$
`	$i\omega\mu\,\vec{H}_T = ik\,\hat{e}_z\times\vec{E}_T$
$\frac{\partial\left(\epsilon\,\vec{E}\right)}{\partial t} = \nabla\times\vec{H}$	$0 = \hat{e}_z\cdot(\nabla_T\times\vec{H}_T)$
`	$-i\omega\epsilon\,\vec{E}_T = -\hat{e}_z\times\nabla_T\psi + ik\,\hat{e}_z\times\vec{H}_T$

Out[∘]=

	TM polarization ($B_z = 0$, $E_z = \psi$)
$\nabla\cdot\, \epsilon\vec{E} = 0$	$\nabla_T\cdot\vec{E}_T + ik\,\psi = 0$
$\nabla\cdot\, \mu\vec{H} = 0$	$\nabla_T\cdot\vec{H}_T = 0$
$-\frac{\partial\left(\mu\,\vec{H}\right)}{\partial t} = \nabla\times\vec{E}$	$0 = \hat{e}_z\cdot(\nabla_T\times\vec{E}_T)$
`	$i\omega\mu\,\vec{H}_T = ?$
$\frac{\partial\left(\epsilon\,\vec{E}\right)}{\partial t} = \nabla\times\vec{H}$	$-i\omega\epsilon\,\psi = \hat{e}_z\cdot(\nabla_T\times\vec{H}_T)$
`	$-i\omega\epsilon\,\vec{E}_T = ?$

Energy flux

Out[◦]=

$$\vec{S} = \tfrac{1}{2}\, \vec{E} \times (\vec{H})^*$$

$$\mathrm{Re}\,\{\hat{e}_z \cdot \vec{S}\} = \langle \tfrac{dP}{da} \rangle$$

Out[◦]=

$$\vec{S} = 1/2\,(\vec{E}_T) \times (\vec{H}_T + \hat{e}_Z H_z) \text{ for a TE mode}$$

See Equation (8.48) .

Note that $\vec{S}_T$ is imaginary and simplify ⇒ the integrated energy flux is

Out[◦]=

$$P = \int_A \vec{S} \cdot \hat{e}_z\, da = \frac{\omega k}{2\,\gamma^4} \int_A |\nabla_T \psi|^2\, da * \Phi$$

where $\Phi = \epsilon$ (TM case) or μ (TE case).

We can simplify the integral still further,

Out[◦]=

$$= \oint_C \psi^* \frac{\partial \psi}{\partial n}\, ds - \int_A \psi^* \nabla^2 \psi\, da$$

by Gauss's theorem

The surface term is zero for either Dirichlet or Neumann boundary conditions, and $\nabla^2\psi = -\gamma^2\psi$;

Out[◦]=

$$P = \frac{1}{2}\left(\frac{\omega}{\omega_\lambda}\right)^2 \sqrt{1 - \omega_\lambda^2/\omega^2} \int_A \psi * \psi\, da * \frac{\Phi}{\sqrt{\mu\epsilon}}$$

Exercise: The field energy per unit length along the z axis is guide is

$$U = \frac{1}{2} \left(\frac{\omega}{\omega_\lambda}\right)^2 \int_A \psi^* \psi \, da * \Phi$$

Group velocity and energy flow and group velocity,

$$\frac{P}{U} = \frac{k}{\mu\epsilon\omega} = \frac{1}{\sqrt{\mu\epsilon}} \sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}} = v_{group}$$

$$v_{group} = \frac{d\omega}{dk} ;$$

$$\text{also} \quad v_g \, v_{phase} = \frac{1}{\mu\epsilon} \approx c^2$$

Attenuation of energy flux

The above results are for perfectly conducting walls.

Now suppose the conductivity σ is finite. There will be ohmic losses $\Longrightarrow$ the energy flux will be attenuated.

Treat the problem by "successive approximations". (Recall Section 8.1.)

In the first approximation, $k_\lambda \approx k_\lambda^{(0)}$ which is real for $\omega > \omega_\lambda$ or pure imaginary for $\omega < \omega_\lambda$.

Now make the correction for finite σ,

$$k_\lambda = k_\lambda^{(0)} + \alpha_\lambda + i \, \beta_\lambda$$

β_λ is the attenuation coefficient.

We will determine β_λ by using Section 8.1; i.e., by conservation of energy

Out[=]=

$$P(z) = P_0\, e^{-2\beta_\lambda z} \implies \beta_\lambda = \frac{-1}{2P}\frac{dP}{dz}$$

Out[=]=

$$-\frac{dP}{dz} = \text{power loss per unit length}$$

$$= \frac{1}{2\sigma\delta} \oint_C |\hat{n} \times \vec{H}|^2\, ds$$

the last line is from Section 8.1.

Example: TE modes of a rectangular waveguide, with δx = a and δy = b, with mode numbers m and n=0.

In[=]:= sc1

Out[=]=

For the TE mode $\lambda = (m0)$ recall

$$H_z = \psi = H_0 \cos\left(\frac{m\pi x}{a}\right)$$

$$\vec{H}_T = \frac{ik}{\gamma^2} \nabla_T \psi$$

Here $\gamma^2 = \mu\epsilon\omega^2 - k^2 = \frac{\pi^2 m^2}{a^2} = \mu\epsilon\,\omega_\lambda^2$

$$k^2 = \mu\epsilon\,\omega^2 - \gamma^2 = \mu\epsilon\,(\omega^2 - \omega_\lambda^2)$$

Calculate $\beta_\lambda = \frac{1}{2P}\left(-\frac{\partial P}{\partial z}\right)$

We have $P = \frac{1}{\sqrt{\mu\epsilon}} \sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}}\; \mathcal{U}$

and $\mathcal{U} = \frac{1}{2}\left(\frac{\omega}{\omega_\lambda}\right)^2 \int_A \psi^* \psi\, da\; \mu$

$\quad da = \int_0^a dx \int_0^b dy$

$$\mathcal{U} = \frac{\mu}{2}\left(\frac{\omega}{\omega_\lambda}\right)^2 H_0^2\, \frac{ab}{2}$$

In[∘]:= sc2

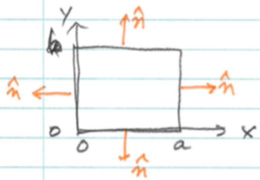
Out[∘]=

And $-\frac{dP}{dz} = \frac{1}{2\sigma\delta} \oint_C |\hat{n}\times\vec{H}|^2 \, ds$

where $\delta = \sqrt{\frac{2}{\mu_0 \omega \sigma}} \propto \frac{1}{\sqrt{\omega}}$

Write $\frac{1}{\delta} = \frac{1}{\delta_\lambda}\frac{\delta_\lambda}{\delta} = \frac{1}{\delta_\lambda}\sqrt{\frac{\omega}{\omega_\lambda}}$ $\qquad \delta_\lambda = \sqrt{\frac{2}{\mu_0 \omega_\lambda \sigma}}$

$\omega_\lambda = \frac{1}{\sqrt{\mu\epsilon}} \frac{m\pi}{a}$



In[∘]:= sc3

Out[∘]=

• Wall at $x=0$: $\hat{n}\times\vec{H} = -\hat{e}_x \times \left[\hat{e}_z \psi + \frac{ik}{\gamma^2}\nabla_t \psi\right]_{x=0}$

$= \hat{e}_y H_0 \qquad \left(\frac{\partial\psi}{\partial x}\right)_{x=0} = 0$

$\Rightarrow \int |\hat{n}\times\vec{H}|^2 ds = H_0^2 \, b$

• Wall at $y=0$: $\hat{n}\times\vec{H} = -\hat{e}_y \times \left[\hat{e}_z \psi + \frac{ik}{\gamma^2}\hat{e}_x \frac{\partial\psi}{\partial x}\right]_{y=0}$

$= -\hat{e}_x H_0 \cos\frac{m\pi x}{a} + \hat{e}_z \frac{ik}{\gamma^2}\left(-\frac{m\pi}{a}\right)\sin\frac{m\pi x}{2} H_0$

$= -H_0\left[\hat{e}_x \cos\frac{m\pi x}{a} + \frac{ik}{\gamma^2}\frac{m\pi}{a}\sin\frac{m\pi x}{a}\right]$

$\Rightarrow \int |\hat{n}\times\vec{H}|^2 ds = H_0^2\left[\frac{a}{2} + \frac{k^2}{\gamma^4}\left(\frac{m\pi}{a}\right)^2 \frac{a}{2}\right]$

$\gamma^2 = \mu\epsilon\,\omega_\lambda^2$ and $k^2 = \mu\epsilon(\omega^2 - \omega_\lambda^2)$

$= \left(\frac{m\pi}{a}\right)^2$

In[∘]:= sc4

Out[∘]=

Result is $-\frac{dP}{dz} = \frac{H_0^2}{\sigma\delta_\lambda}\sqrt{\frac{\omega}{\omega_\lambda}}\left[b + a\frac{\omega^2}{\omega_\lambda^2}\right]$

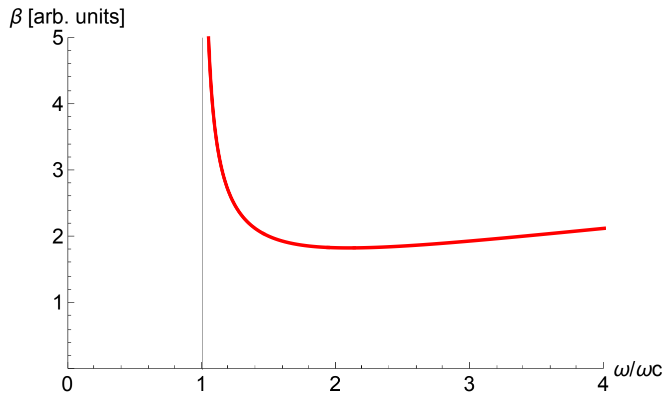
$\Rightarrow$

$\beta_\lambda = \frac{2\sqrt{\epsilon/\mu}}{\sigma\delta_\lambda}\sqrt{\frac{\omega}{\omega_\lambda}}\frac{1}{\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}}\left[\frac{1}{b} + \frac{1}{a}\frac{\omega_\lambda^2}{\omega^2}\right]$

(many factors cancel to get here)

In[ ]:=
```
(* Plot in Figure 8.6 *)
(* x = ω/ωc *)
β = Sqrt[x / (1 - 1 / x^2)] * (1 / b + 1 / a * (1 / x^2));
fun[x_] = β /. {a → 2, b → 1} // Simplify;
Show[
 Plot[fun[x], {x, 1, 4}, PlotRange → {{0, 4}, {0, 5}},
  AxesLabel → {"ω/ωc", "β [arb. units]"},
  PlotStyle → {Red, AbsoluteThickness[3]}, BaseStyle → 18],
 Graphics[ { Line[{{1, 0}, {1, 5}}] }]]
```

Out[ ]=



In[ ]:=
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fig8p6
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Out[ ]=

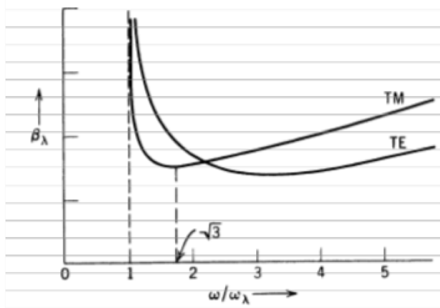

Figure 8.6 Attenuation constant β_λ as a function of frequency for typical TE and TM modes. For TM modes the minimum attenuation occurs at $\omega/\omega_\lambda = \sqrt{3}$, regardless of cross-sectional shape.