ente

Resonant Cavities
Section 8.7

In[]:= w0

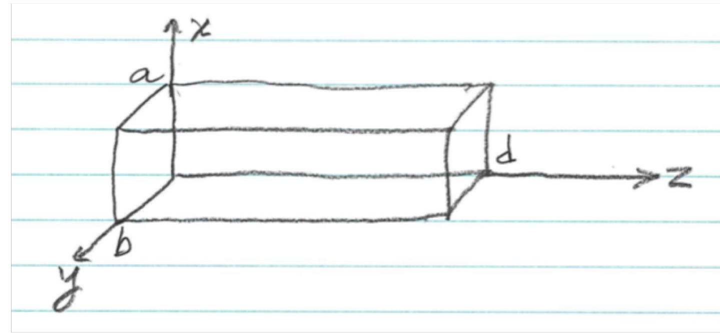
Out[]=

Two microwave cavities *(left)* from 1955, each attached by waveguide to a reflex klystron *(right)* a vacuum tube used to generate microwaves. The cavities serve as resonators (tank circuits) to determine the frequency of the oscillators

Waveguides

w1

Out[]=



but we never worried about z = 0 and z = d.
It should be OK if $d \gg \lambda$,

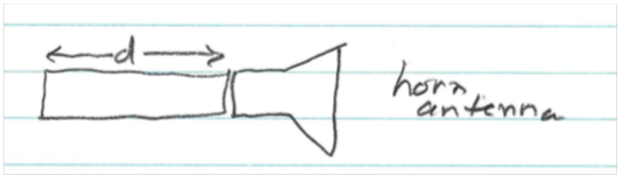
For f = 30 GHz microwaves,

$$\lambda = \frac{c}{f} = \frac{3\times 10^8\ m/s}{3\times 10^{10}\ /s} = 0.01\ \mathrm{m} = 1\ \mathrm{cm}$$

Or, it should be OK if the end is attached to something else

w2

Out[•]=



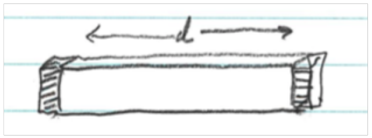
Resonators

An enclosed volume

Put conducting caps on the ends of a waveguide

w3

Out[•]=



The field equations in the volume are the same as for the waveguide; but now there are two more boundary conditions,
at z = 0 and z = d.

For perfectly conducting caps,

■ $\vec{E}_{\text{tangential}} = 0$ at z = 0 and z = d;
i.e., $\vec{E}_T = 0$ at ends.

■ $B_{\text{normal}} = 0$ at z = 0 and z = d;
i.e., $B_z = 0$ at ends.

Write

$$\vec{F}(\vec{x},t) = \vec{F}(x,y,z)\, e^{-i\omega t}$$

∎ For traveling waves, as in a waveguide,

$$\vec{F}(x,y,z) = \vec{F}(x,y) \times \{ e^{ikz} \text{ or } e^{-ikz} \}$$

and linear combinations.

∎ For standing waves, as in a cavity resonator,

$$\vec{F}(x,y,z) = \vec{F}(x,y) \times \{ \cos(kz) \text{ or } \sin(kz) \}$$

and linear combinations;

but here the boundary conditions at $z = 0$ and d must be satisfied:

∎∎ Dirichlet boundary conditions

($\vec{F} = 0$ at $z = 0$ and d)

$$\Longrightarrow \sin(kz) \text{ and } k = \frac{p\pi}{d}$$

where p is an integer ;

∎∎ Neumann boundary conditions

($\partial\vec{F}/\partial z = 0$ at $z = 0$ and d)

$$\Longrightarrow \cos(kz) \text{ and } k = \frac{p\pi}{d} \ .$$

TM fields and TE fields

TM fields ($B_z = 0$)	**TE fields** ($E_z = 0$)
$E_z = \psi(x,y)\cos(p\pi z/d)$	$B_z = \psi(x,y)\sin(p\pi z/d)$
$\vec{E}_T = \frac{-p\pi/d}{\gamma^2}\sin(\frac{p\pi z}{d})\,\nabla_T\psi$	$\vec{E}_T = \frac{-i\omega\mu}{\gamma^2}\sin(\frac{p\pi z}{d})\,\hat{e}_z \times \nabla_T\psi$
$\vec{H}_T = \frac{i\omega\epsilon}{\gamma^2}\cos(\frac{p\pi z}{d})\,\hat{e}_z\times\nabla_T\psi$	$\vec{H}_T = \frac{p\pi/d}{\gamma^2}\cos(\frac{p\pi z}{d})\,\nabla_T\psi$

where $k = p\pi/d$ and $\gamma^2 = \mu\epsilon\omega^2 - (p\pi/d)^2$. The boundary conditions at $z = 0$ and d are obeyed. Also

$$(\nabla_T^2 + \gamma^2)\,\psi = 0$$

$$\text{and } \psi\,|_S = 0 \text{ (TM) or } \hat{n}\bullet\nabla_T\,\psi\,|_S = 0 \text{ (TE)}$$

Mode numbers

(analogous to quantum numbers)

Given $p \in \{0,1,2,3,\ldots\}$ there will be two mode numbers $\lambda = \{m, n\}$ and an eigenfrequency $\gamma_\lambda(p)$ and eigenfunction $\psi_\lambda(p; x,y)$. The frequency of the mode — $\omega_\lambda(p)$ — is given by

$$\gamma^2 = \mu\epsilon\omega^2 - k^2 = \mu\epsilon\omega^2 - (p\pi/d)^2$$

$$\omega_\lambda(p)^2 = \frac{1}{\mu\epsilon}\left[\,\gamma_\lambda(p)^2 + (\frac{p\pi}{d})^2\,\right]$$

Resonant frequencies of the cavity

$$\{\,\omega_\lambda(p) \text{ for } \lambda \in \Lambda \text{ and } p \in \{1,2,3,\ldots\}$$

a discrete set of eigenfrequencies, i.e., such that all the fields $\propto e^{-i\omega t}$. Choose the size and shape of the boundary such that the frequency of operation is near one of the eigenfrequencies and well away from the others.

The right circular cylinder

There are two parameters:
inner radius R and length d.

First consider the TM modes:

- TM modes;

so $E_z = \psi(\rho,\phi)$ where ρ and ϕ are plane polar coordinates.

- $p \in \{ 0, 1, 2, 3, \ldots\}$

The field equation is

$$(\nabla_T^2 + \gamma^2)\, \psi = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \gamma^2\,\psi = 0$$

and the boundary condition is

$$\psi\,|_S = \psi(R,\phi) = 0$$

The solution is

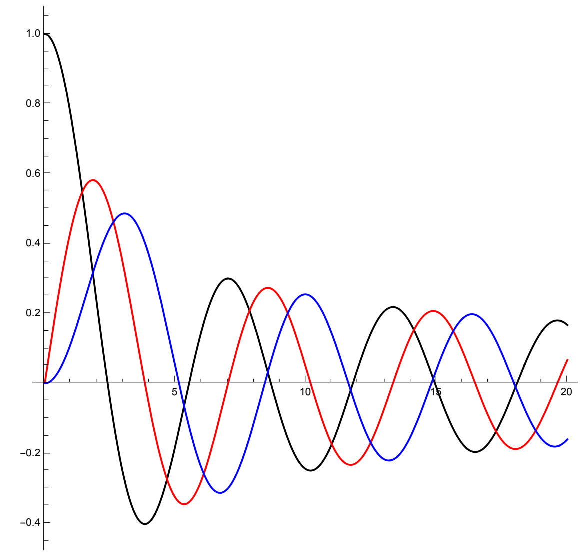
$$\psi(\rho,\phi) = E_0\, J_m(\gamma_{mn}\rho)\, e^{\pm im\phi}$$

where

$\gamma_{mn}R = x_{mn}$ and $J_m(x_{mn}) = 0$

x_{mn} = the n^{th} zero of $J_m(\xi)$

```
Plot[{BesselJ[0, z], BesselJ[1, z], BesselJ[2, z]}, {z, 0, 20},
 PlotStyle → {Black, Red, Blue}, ImageSize → 480, AspectRatio → 1]
```



Zeros of the Bessel functions

```
r0 = {{"m=0", "m=1", "m=2", "m=3"}};
values = Table[ SetPrecision[BesselJZero[m, n], 3],
   {n, 1, 4}, {m, 0, 3}];
tbl = Style[Join[r0, values] // TableForm, ff]
```

m=0	m=1	m=2	m=3
2.41	3.83	5.14	6.38
5.52	7.01	8.42	9.76
8.65	10.2	11.6	13.0
11.8	13.3	14.8	16.2

The resonant frequencies are

$$\omega_{mn}(p) = \frac{1}{\sqrt{\mu\epsilon}} \left[\frac{x_{mn}^2}{R^2} + \frac{p^2 \pi^2}{d^2} \right]^{1/2}$$

The lowest TM mode has

{p,m,n} = {0, 0, 1};

$$\omega_{01}(0) = \frac{1}{\sqrt{\mu\epsilon}} \frac{2.41}{R}$$

For $\mu = \mu_0$ and $\epsilon = \epsilon_0$ and R = 3 cm,

$\omega_{01}(0) = 2\pi\times(3.84\text{ GHz})$ and $\lambda = 7.82$ cm

$\omega_{01}(0)$ does not depend on d,

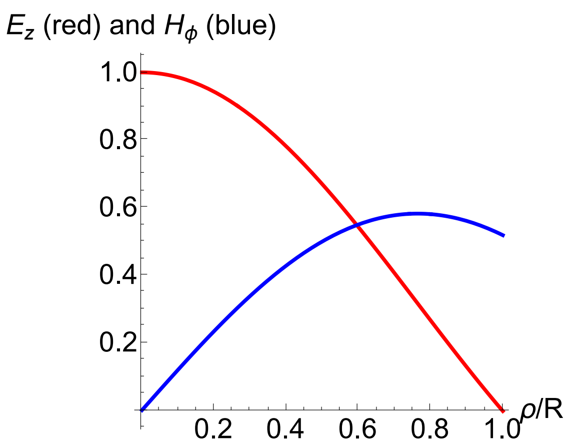
so tuning is not possible.

The fields inside the cavity for mode TM(0,1,0)

$$E_z = E_0\, J_0(x_{01}\, \rho/R)\, e^{-i\omega t}$$

$$H_\phi = -i \sqrt{\epsilon/\mu}\, E_0\, J_1(x_{01}\, \rho/R)\, e^{-i\omega t}$$

```
Plot[ { BesselJ[0, 2.41 * x], BesselJ[1, 2.41 * x]},
 {x, 0, 1},
 PlotStyle → {{Thickness[0.01], Red}, {Thickness[0.01], Blue}},
 BaseStyle → 24, AxesLabel → {"ρ/R", "Ez (red) and Hϕ (blue)"},
 ImageSize → 480, AspectRatio → 1]
```



TE modes

$H_z = \psi(x,y)\sin\left(\frac{p\pi z}{d}\right)$ $(p = 1\,2\,3\ldots)$

Again $\psi(\rho,\phi) = H_0 J_m(\gamma_{mn}\rho)\, e^{\pm i m\phi}$

But $\gamma_{mn} = \frac{x'_{mn}}{R}$ where $J'_m(x'_{mn}) = 0$

(x'_{mn} = n^{th} zero of J'_m)

Then

$$\omega_{mn}(p) = \frac{1}{\sqrt{\mu\epsilon}}\left(\frac{x'^2_{mn}}{R^2} + \frac{p^2\pi^2}{d^2}\right)^{1/2}$$

$p = 1\;2\;3\ldots.$

The lowest TE mode :

$\{p, m, n\} = \{1, 1, 1\}$

tunable ↓

$$\omega_{11}(1) = \frac{1.841}{\sqrt{\mu\epsilon}\,R}\left[1 + 2.912\,\frac{R^2}{d^2}\right]^{1/2}$$

$$\psi_{111} = H_0\, J_1\left(\frac{1.841\rho}{R}\right)\cos\phi\,\sin\left(\frac{\pi z}{d}\right) e^{-i\omega t}$$