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Cavity Resonators
Sections 8.7 and 8.8

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Two microwave cavities *(left)* from 1955, each attached by waveguide to a reflex klystron *(right)* a vacuum tube used to generate microwaves. The cavities serve as resonators (tank circuits) to determine the frequency of the oscillators

Fields in a Cavity Resonator

$\vec{F}(\vec{x},t) = \vec{F}(x,y,z)\, e^{-i\omega t}$;

or, for a circular cylinder, $\{x,y\} \rightarrow \{\rho,\phi\}$

TM fields ($B_z = 0$)	**TE fields** ($E_z = 0$)
$E_z = \psi(x,y)\cos(p\pi z/d)$	$B_z = \psi(x,y)\sin(p\pi z/d)$
$\vec{E}_T = \frac{-p\pi/d}{\gamma^2}\sin(\frac{p\pi z}{d})\,\nabla_T\psi$	$\vec{E}_T = \frac{-i\omega\mu}{\gamma^2}\sin(\frac{p\pi z}{d})\,\hat{e}_z \times \nabla_T\psi$
$\vec{H}_T = \frac{i\omega\epsilon}{\gamma^2}\cos(\frac{p\pi z}{d})\,\hat{e}_z\times\nabla_T\psi$	$\vec{H}_T = \frac{p\pi/d}{\gamma^2}\cos(\frac{p\pi z}{d})\,\nabla_T\psi$

where $k = p\pi/d$ and $\gamma^2 = \mu\epsilon\omega^2 - (p\pi/d)^2$;

p is an integer, e.g., 0 or 1.

The boundary conditions at z = 0 and d are obeyed.

Now solve $(\nabla_T^2 + \gamma^2)\,\psi = 0$

with $\psi\,|_S = 0$ (TM) or $\hat{n}\bullet\nabla_T\,\psi\,|_S = 0$ (TE)

The right circular cylinder

Parameters: inner radius R and length d.

Bessel functions J_m(x)

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Plot[{BesselJ[0, z], BesselJ[1, z], BesselJ[2, z]}, {z, 0, 20},
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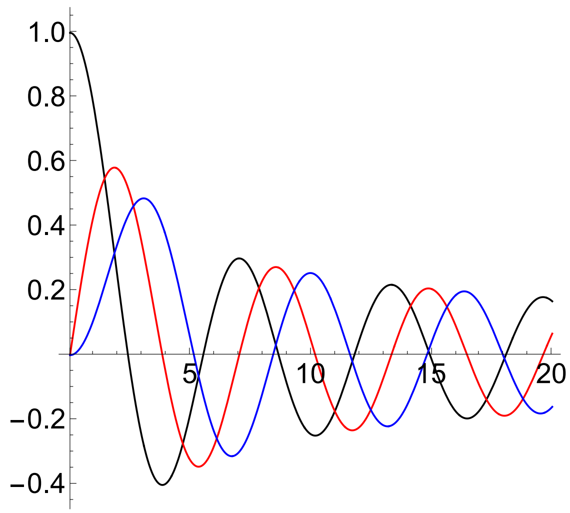


TM modes (B_z=0)	TE modes (E_z=0)
$E_z = \psi(\rho,\phi)$ $= E_0\, J_m(\xi\rho/R)\, e^{im\phi}$	$H_z = \psi(\rho,\phi)$ $= H_0\, J_m(\xi\rho/R)\, e^{im\phi}$
$\xi = x_{mn}$ = zero of J_m	$\xi = x'_{mn}$ = zero of J'_m

The resonant frequencies are

$$\omega_{\rm mn}(\mathrm{p}) = \frac{1}{\sqrt{\mu\epsilon}}\ [\ \frac{\xi_{\rm mn}^2}{R^2} + \frac{p^2\pi^2}{d^2}\ ]^{1/2}$$

Jackson has calculated the fields.
Homework: What are the surface charge density Σ, and the surface current density $\vec{K}$, on S and on the ends E_1 and E_2 ?

Section 8.8 — Power loss in a cavity; Q

We have considered surfaces of *perfect conductors*. For metals the conductivity σ is finite, and the "surface current" extends into a thin layer of the metal;

skin depth $\delta = \sqrt{2/\sigma\omega}$.

$\Longrightarrow$ ohmic energy loss

Q of the cavity

$$Q = \omega_0 \frac{\text{Stored energy}}{\text{Power loss}}$$

where ω_0 is the resonant frequency.

Furthermore, the frequency response is not a delta function; it is smeared out by an amount that depends on Q.

Proof

$$\frac{dU}{dt} = -\frac{\omega_0}{Q} U \; ; \; \therefore \; U(t) = U(0)\, e^{-\omega_0 t/Q}$$

$$F(t) = F_0\, e^{-\omega_0 t/(2Q)}\, e^{-i(\omega_0 + \Delta\omega) t}$$

$$F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{F}(\omega)\, e^{-i\omega t}\, d\omega$$

$$\tilde{F}(\omega) = \frac{1}{\sqrt{\mu\epsilon}} \int_{-\infty}^{\infty} [F_0\, e^{-\omega_0 t/(2Q)}\, e^{-i(\omega_0 + \Delta w) t}]\, e^{i\omega t}\, dt$$

from which we obtain the resonant line shape

$$|\tilde{F}(\omega)|^2 \propto \frac{1}{(\omega - \omega_0 - \Delta\omega)^2 + \omega_0^2/(2Q)^2}$$

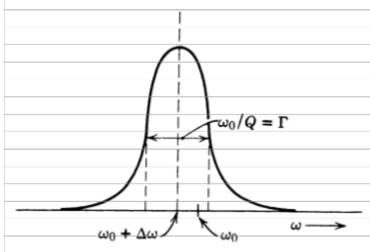


Figure 8.8 Resonance line shape. The full width Γ at half-maximum (of the power) is equal to the unperturbed frequency ω_0 divided by the Q of the cavity.

$$Q = \frac{\omega_0}{\delta\omega} = \frac{\omega_0}{\Gamma}$$

Jackson: “Q values of several hundreds or thousands are common for microwave circuits.”

Now, calculate Q

Same method as we used to calculate attenuation of energy flux in a waveguide

…

- First calculate stored energy (averaged over time)
- Then calculate the power loss in the walls
- We’ll do the calculations for cylindrical cavities; for the modes TM(λ,p) and TE(λ,p)

$$U = \frac{d}{4}\,\Phi\left[1+\left(\frac{p\pi}{\gamma d}\right)^2\right]\int_A |\psi|^2\, da$$

Out[•]=

$$\Phi = \epsilon(1+\delta_{p0}) \text{ for TM, or } \mu \text{ for TE}$$

$$P_{loss} = \frac{1}{2\pi\delta}\left\{\int_S da\, |\hat{n}\times\vec{H}|^2 + 2\int_{E1+E2} da\, |\hat{n}\times\vec{H}|^2\right\}$$

Example
The cavity mode TM(0,1,0)

p = 0 $\Longrightarrow$ cos(pπz/d) = 1

n = 1 $\Longrightarrow$ the first zero of $J_0(\xi)$ is $\xi = 2.405$

m = 0 $\Longrightarrow$ $e^{im\phi} \to 1$

$$E_z = E_0\; J_0(\xi\rho/R)\, e^{-i\omega t}$$

$$H_\phi = -i\sqrt{\epsilon/\mu}\; E_0\; J_1(\xi\rho/R)\, e^{-i\omega t}$$

In[◦]:= scanc1

Out[◦]=

$$\text{TM}\,(\overset{p}{0},\overset{n}{1},\overset{m}{0})\ \text{mode}\ ;\quad \omega_{010} = \frac{x_{01}}{\sqrt{\mu\epsilon}\,R}$$

$$E_z = E_0 J_0\left(\frac{\xi\rho}{R}\right) \quad \text{where}\quad \xi = x_{01} = 2.405$$

$$\vec{E}_T = 0$$

$$\vec{H}_T = \hat{\phi} H_\phi \ ; \quad H_\phi = -i\sqrt{\frac{\epsilon}{\mu}}\, E_0 J_1\left(\frac{\xi\rho}{R}\right)$$

In[◦]:= scanc2

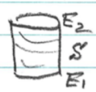
Out[◦]=

<u>Stored Energy</u> $\quad U = \frac{d}{4}\cdot 2\epsilon \cdot \int_A |\psi|^2 da$

$$U = \epsilon E_0^2\, d R^2\, (0.135)$$

<u>Power Loss</u>

$$P_{loss} = \frac{1}{2\sigma\delta}\left\{ \int_S |\hat{n}\times\vec{H}|^2 da + 2\int_{E_1 E_2} |\hat{n}\times\vec{H}|^2 da \right\}$$



For S, $\hat{n} = \hat{\rho}$ and $\rho = R$

For E_1, $\hat{n} = -\hat{e}_z$ and $z = 0$

$$P_{loss} = \frac{\epsilon}{\mu} E_0^2 \frac{1}{2\sigma\delta} (0.269)\, 2\pi R\, (d + 2R)$$

In[◦]:= scanc3

Out[◦]=

<u>Q</u> $\quad = \omega \frac{U}{P_{loss}}$

After some algebra ...

$$Q = \frac{d}{\delta}\left(\frac{\mu}{\mu_c}\right) \frac{R^2}{2R^2 + Rd} \qquad (?)$$

Very large values of Q are possible!

Skin depth vs frequency for some metals at room temperature (en.wikipedia.org/wiki/Skin_effect)