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Theory of Radiation
Chapter 9

We have studied waves and how they propagate...

- in free space
- in dielectrics
- in metals and plasmas
- in waveguides

Now, *how are these waves created*?

Chapter 9 is about the theory of radiation.

This theory is the prerequisite for Chapter 10 — scattering and diffraction.

Fields and radiation of a localized oscillating source

Out[ ]=

$$\rho(\vec{x},t) = \rho(\vec{x})\, e^{-i\omega t}$$
$$\vec{J}(\vec{x},t) = \vec{J}(\vec{x})\, e^{-i\omega t}$$
$$\text{with} \quad i\omega\rho = \nabla\cdot\vec{J}$$

The real part is the physical quantity.

These are the sources.

What are the fields?

Well, first, we'll calculate the potentials.

In the Lorenz gauge, recall

Out[ ]=

$$\vec{B} = \nabla\times\vec{A} \quad \text{and} \quad \vec{E} = -\nabla\Phi - \frac{\partial\vec{A}}{\partial t}$$
$$\text{with} \quad \nabla\cdot A + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0$$

Recall the retarded Green' s function for the wave equation

Out[•]=

$$\vec{A}(\vec{x},t) = \frac{\mu 0}{4\pi} \int d^3x' \int dt' \frac{J(\vec{x}', t')}{\left|\vec{x} - \vec{x}'\right|}$$

$$\delta(t' - t + |\vec{x} - \vec{x}'|/c)$$

For a harmonic source, $\vec{J}\left(\vec{x}'\right) e^{-i\omega t'}$ ...

$$\vec{A}(\vec{x},t) = \vec{A}(\vec{x})\, e^{-i\omega t}$$

Out[•]=

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}\left(\vec{x}'\right)}{\left|\vec{x}-\vec{x}'\right|} e^{ik\left|\vec{x}-\vec{x}'\right|} \quad (\bigstar)$$

where $k = \omega/c$ (wave number)

$$\vec{H} = \frac{1}{\mu_0} \nabla\times \vec{A}$$

Out[•]=

$$\vec{E} = \frac{iZ_0}{k} \nabla\times \vec{H} \quad \textit{outside the source}$$

$$Z_0 = \sqrt{\mu_0/\epsilon_0} = \mu_0\, c = 377\ \text{ohm}$$

In principle this solves the problem:

Given $\vec{J}(\vec{x})\, e^{-i\omega t}$, do the integral ($\star$).

The approximate solution for a small source

Let d = "the source dimension";
a characteristic size.

Let λ = the wavelength of the
radiated waves; $\lambda \equiv 2\pi c/\omega$.

Assume d $\ll \lambda$.

Think of examples ...

Out[ ]=

λ	example	limit d $\ll$
500 nm	yellow light	5000 A
3 GHz	MW oven	10 cm
1 MHz	AM radio	300 m

Zones

Assume d $\ll \lambda$.

Out[ ]=

name	(alternative)	r range
Near zone	(static)	d $\ll$ r $\ll \lambda$
Intermediate	(induction)	d $\ll$ r ~ λ
Far zone	(radiation)	d $\ll \lambda \ll$ r

The most interesting case is the radiation fields.

In the far zone, $\vec{E}(\vec{x})$ and $\vec{B}(\vec{x})$ ~ O(1/r) for large r.

That implies $\vec{S}(\vec{x}) \sim O(1/r^2)$.
This is the "inverse square law of energy flux" which is required by conservation of energy.

The near zone

The factor e^{ikR} in the integral (★):

$e^{ik|\vec{x}-\vec{x}'|}$ where $k = 2\pi/\lambda$

and $|\vec{x} - \vec{x}'| = \sqrt{r^2 + (r')^2 - 2\, r\, r' \cos(\gamma)}$.

In the near zone we can approximate

$e^{ik|\vec{x}-\vec{x}'|} \approx 1$. Then

Out[•]=

$$A(x,t) \approx \frac{\mu_0}{4\pi} \int \frac{J(x')\, d^3 x'}{|\, x - x'\,|}\; e^{-i\omega t}$$

The fields are "quasi-stationary"; oscillating in time, but static in space.

The Far Zone ; r ≫ λ ; kr = 2πr/λ ≫ 2π.

Now we use this approximation:

Out[•]=

$$|\vec{x}-\vec{x}'| = \sqrt{r^2 + (r')^2 - 2\, r\, r' \cos(\gamma)}$$

$$\approx r - r' \cos(\gamma) + O(r'^2/r)$$

We'll neglect the last term $O(d^2/r)$ compared to *d*. Furthermore, we'll approximate

$$\frac{1}{|\vec{x}-\vec{x}'|} \approx \frac{1}{r} .$$

So the result is

Out[•]=

$$\vec{A}(\vec{x},t) = \frac{e^{ikr}\, e^{-i\omega t}}{r}\; \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}')\, e^{-ik\,\hat{n}\cdot\vec{x}'}\, d^3x'$$

Comments

- $\frac{1}{r}\, e^{i(\mathrm{kr}-\omega \mathrm{t})}$ is an outgoing spherical wave.

Note the "inverse square law".

- The integral depends only on the angles of $\hat{n}$;

— observation position = $\vec{x}$ = r $\hat{n}$;

— $\hat{n} = \hat{e}_x \sin\theta \cos\phi + \hat{e}_y \sin\theta \sin\phi + \hat{e}_z \cos\theta$;

(eventually we'll pick the Cartesian directions)

Out[∘]=

$$\vec{A}(\vec{x},t) = \frac{e^{ikr}\, e^{-iwt}}{r} \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}')\, e^{-ik\,\hat{n}\cdot\vec{x}'}\, d^3x'$$

- We don't need the scalar potential in the far zone, because we can use Faraday's law to calculate $\vec{E}$.

Electric Dipole Fields and Radiation (Section 9.2)

The potential in the far zone is

In[∘]:=

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Style["A(x,t) = e^ikr e^-iwt / r  μ0/(4π)  ∫ J(x') e^-ik n·x' d³x' ", ff, 32]
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Out[∘]=

$$\vec{A}(\vec{x},t) = \frac{e^{ikr}\, e^{-iwt}}{r} \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}')\, e^{-ik\,\hat{n}\cdot\vec{x}'}\, d^3x'$$

Now k $\hat{n} \bullet \vec{x}$' is $O\left(\frac{2\pi d}{\lambda}\right)$, small in far zone.

We can expand exp[−ik$\hat{n}\bullet\vec{x}$'] in powers of k$\hat{n}\bullet\vec{x}$'.

The result is the *multipole expansion for radiation.*

The dominant term is the *electric dipole approximation*;

approximate exp[−ik$\hat{n}\bullet\vec{x}$'] ≈ 1.

Out[∘]=

$$\vec{A}(\vec{x}) = \frac{e^{ikr}}{r} \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}')\, d^3x'$$

Rewrite the result

Out[=]=

$\int J\, d^3x' = -\int \vec{x}'(\nabla'\cdot\vec{J})\, d^3x'$	by Gauss's theorem
$= -i\omega \int \vec{x}'\, \rho(\vec{x}')\, d^3x'$	by the continuity equation
$= -i\omega\, \vec{p}$	$\vec{p}$ = electric dipole moment

Here $\vec{p}$ is the electric dipole moment of the charge density $\rho(\vec{x})$.

Out[=]=

$$\vec{A}(\vec{x}) = \frac{-i\,\mu_0\,\omega}{4\,\pi}\,\vec{p}\,\frac{e^{ikr}}{r}$$

$$\vec{p} = \int \vec{x}'\,\rho(\vec{x}')\,d^3x'$$

Calculate $\vec{E}$ and $\vec{B}$ in the radiation zone, from the electric dipole potential.

The results are (exercise)

Out[=]=

Electric dipole radiation
$\vec{H} = \frac{ck^2}{4\,\pi}\,(\hat{n}\times\vec{p})\,\frac{e^{ikr}}{r}\,e^{-i\omega t}$
$\vec{E} = Z_0\,\vec{H}\times\hat{n}$

“showing the typical behavior of radiation fields.”

Power radiated per unit solid angle by an oscillating electric dipole

and $\hat{n} \cdot \vec{S}$ dA = dP; dA = r^2 dΩ.

The ***time-averaged*** differential power is

Out[◦]=
$$\frac{dP}{d\Omega} = \frac{1}{2} \mathrm{Re}[\, r^2\, \hat{n} \bullet \vec{E} \times \vec{H}^* \,]$$

Exercise.

Show, from the equations for $\vec{H}$ and $\vec{E}$ in the far zone,

Out[◦]=
$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\,\pi^2}\, k^4\, |(\hat{n}\times\vec{p})\times\hat{n}|^2$$

- Footnote about ***polarization***.
- If $\vec{p}$ is real then

Out[◦]=
$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\,\pi^2}\, k^4\, p^2 \sin^2\theta$$

where θ is the angle between $\vec{p}$ and $\hat{n}$;
dP/dΩ $\propto \sin^2\theta$ is called “typical dipole”;
(draw a picture)

The total power; (note $\int \sin^2\theta\ d\Omega = \frac{8\pi}{3}$)

Out[◦]=
$$P = \frac{c^2 Z_0}{12\,\pi}\, k^4\, p^2$$