enter

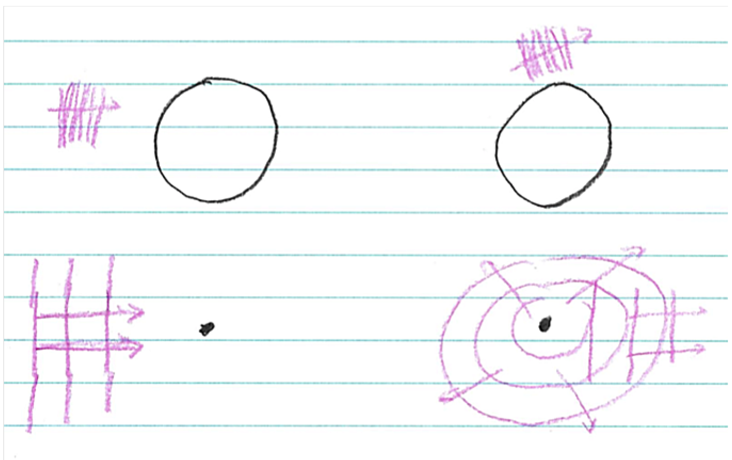
Scattering and Diffraction
Chapter 10

Chapter 9 is about the theory of radiation.

Chapter 10 — Scattering and Diffraction — makes use of the theory of radiation.

What is the difference between reflection and scattering?

In[•]:= scanC



Radiation by an oscillating electric dipole (Section 9.2)

For $d \ll \lambda$, electric dipole radiation is usually the dominant form of radiation.

Consider a charge distribution for which the electric dipole moment oscillates harmonically with frequency ω,

$$\vec{p}(t) = \vec{p}\, e^{-i\omega t}$$

Then the vector potential in the far zone is

$$\vec{A}(\vec{x}) = \frac{-i\,\mu_0\,\omega}{4\,\pi}\,\vec{p}\,\frac{e^{ikr}}{r}\,e^{-i\omega t}$$

$$\vec{p} = \int \vec{x}'\,\rho(\vec{x}')\,d^3x'$$

The radiation fields

(homework problem 13-1)

E1 radiation	far zone
$\vec{H} = \frac{1}{\mu_0} \nabla \times \vec{A}$	$\sim \frac{ck^2}{4\pi} (\hat{n} \times \vec{p}) \frac{e^{ikr}}{r} e^{-i\omega t}$
$\vec{E} = \frac{iZ_0}{k} \nabla \times \vec{H}$	$\sim Z_0 \vec{H} \times \hat{n}$

where $Z_0 = \sqrt{\mu_0 / \epsilon_0} = \mu_0\, c = 377$ ohm.

Section 10.1
Scattering for long wavelengths

We start with an incident plane wave

$\mathbf{E_{inc}} = \boldsymbol{\epsilon_0} E_0 e^{ik\, \mathbf{n_0} \cdot \mathbf{x}}$	$\boldsymbol{\epsilon_0}$ = pol. vector
	$\mathbf{n_0}$ = inc. direction
$\mathbf{H_{inc}} = \mathbf{n_0} \times \mathbf{E_{inc}} / Z_0$	

where $k = \omega / c$.

(Factor $e^{-i\omega t}$ is understood.)

∎ These fields of the incident wave *polarize the object*

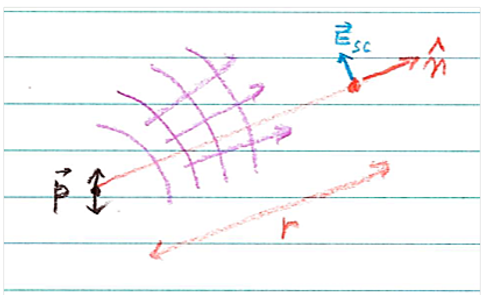
$\Longrightarrow$ electric dipole moment $\vec{p}(t)$

and perhaps magnetic dipole moment $\vec{m}(t)$

■ These sources radiate outgoing waves.
In the radiation zone

$\mathbf{E_{sc}} = \dfrac{k^2 e^{ikr}}{4\pi\epsilon_0 r} [(\mathbf{n}\times\mathbf{p})\times\mathbf{n} - \mathbf{n}\times\mathbf{m}/c]$
$\mathbf{n}$ = direction of observation
r = distance from the object
$\mathbf{H_{sc}} = \mathbf{n} \times \mathbf{E_{sc}}/Z_0$

scanA



The radiated power

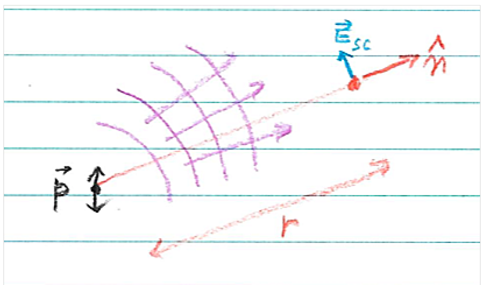
$\dfrac{dP}{d\Omega} = r^2\, \mathbf{n}\bullet \{\mathrm{Re}\, \mathbf{E}_{sc}\} \times \{\mathrm{Re}\, \mathbf{H}_{sc}\}$
or, $\dfrac{dP}{d\Omega} = \dfrac{1}{2} r^2\, \mathrm{Re}\{ \mathbf{n}\bullet \mathbf{E_{sc}} \times \mathbf{H_{sc}}^* \}$
or, $\dfrac{dP}{d\Omega} = \dfrac{1}{2} r^2\, \vert \boldsymbol{\epsilon}^*\bullet \mathbf{E_{sc}} \vert^2 \dfrac{1}{Z_0}$
and inc flux $= \dfrac{1}{2 Z_0} \vert \boldsymbol{\epsilon_0}^*\bullet \mathbf{E_{inc}} \vert^2$

The differential scattering cross section

$$\frac{d\sigma}{d\Omega}[\mathbf{n},\boldsymbol{\epsilon};\mathbf{n}_0,\boldsymbol{\epsilon}_0] = \frac{dP/d\Omega}{\text{inc. power flux}}$$

$$\frac{d\sigma}{d\Omega}[\mathbf{n},\boldsymbol{\epsilon};\mathbf{n}_0,\boldsymbol{\epsilon}_0] = \frac{r^2\,|\,\boldsymbol{\epsilon}^*\bullet\mathbf{E}_{sc}\,|^2}{|\,\boldsymbol{\epsilon}_0^*\bullet\mathbf{E}_{inc}\,|^2}$$

In[]:= scanA



Result for dipole radiation

$$\frac{d\sigma}{d\Omega}[\mathbf{n},\boldsymbol{\epsilon};\mathbf{n}_0,\boldsymbol{\epsilon}_0] = \quad ,$$

$$= \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\,|\boldsymbol{\epsilon}^*\bullet\mathbf{p} + (\mathbf{n}\times\boldsymbol{\epsilon}^*)\bullet\mathbf{m}/c\,|^2$$

In[]:= scanB

Out[]=

$$\hat{\epsilon}^*\cdot[(\hat{n}\times\vec{p})\times\hat{n}] = \hat{\epsilon}^*\cdot[\vec{p}-\hat{n}(\hat{n}\cdot\vec{p})] = \hat{\epsilon}^*\cdot\vec{p}$$

Comment.

The moments $\vec{p}$ and $\vec{m}$ will depend on $\vec{E}_{\text{inc}}$, because these are moments of the distribution *induced by the incident wave*.

So $d\sigma/d\Omega$ will not depend on E_0, and will depend on the incident polarization $\hat{\boldsymbol{\epsilon}}_0$.

To proceed we need to consider specific examples of the scattering object.

(B) Scattering by a small dielectric sphere

Parameters: radius a, permeability $\mu = \mu_0$, relative permittivity $\epsilon_r = \epsilon(\omega)/\epsilon_0$.

Assuming the frequency is small enough to justify the quasi-static approximation for the dipole moment,

$$\mathbf{p} = 4\pi\epsilon_0\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right) a^3\, \mathbf{E_{inc}}$$

$$\frac{d\sigma}{d\Omega} = k^4\, a^6\, \left|\frac{\epsilon_r - 1}{\epsilon_r + 2}\right|^2 \left|\boldsymbol{\epsilon}^* \bullet \boldsymbol{\epsilon}_0\right|^2$$

Interesting feature: the scattered radiation is linearly polarized in the plane spanned by $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{\epsilon}}_0$.

Assume the incident radiation is unpolarized (that's the usual thing). Then what is the dependence of the cross section on $\hat{\boldsymbol{\epsilon}}$?

$$\frac{d\sigma_{\parallel}}{d\Omega} = \frac{1}{2}\, k^4 a^6 \left|\frac{\epsilon_r - 1}{\epsilon_r + 2}\right|^2 \cos^2\theta$$

$$\frac{d\sigma_{\perp}}{d\Omega} = \frac{1}{2}\, k^4 a^6 \left|\frac{\epsilon_r - 1}{\epsilon_r + 2}\right|^2$$

where $\hat{\boldsymbol{\epsilon}}$ is $\begin{cases} \parallel & \text{to the scattering plane} \\ \perp & \text{to the} \quad \text{"} \end{cases}$

In[ ]:= fig10p1

Out[ ]=

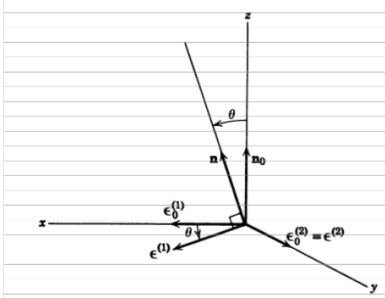


Figure 10.1 Polarization and propagation vectors for the incident and scattered radiation.

Results and Figure 10.2

$$\Pi(\theta) = \frac{\sigma\perp - \sigma\parallel}{\sigma\perp + \sigma\parallel}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = k^4 a^6 |(\epsilon_r - 1)/(\epsilon_r + 2)|^2 (1+\cos^2\theta)/2$$

$$\sigma = \frac{8\pi}{3} k^4 a^6 |..|^2$$

fig10p2

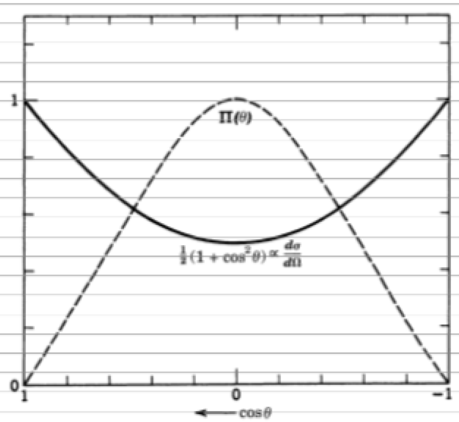


Figure 10.2 Differential scattering cross section (10.10) and the polarization of scattered radiation (10.9) for a small dielectric sphere (dipole approximation).

What can we say about this?

scan1

The differential cross section (10.10) and the polarization of the scattered radiation (10.9) are shown as functions of cos θ in Fig. 10.2. The polarization $\Pi(\theta)$ has its maximum at $\theta = \pi/2$. At this angle the scattered radiation is 100% linearly polarized perpendicular to the scattering plane, and for an appreciable range of angles on either side of $\theta = \pi/2$ is quite significantly polarized. The polarization characteristics of the blue sky are an illustration of this phenomenon, and are, in fact, the motivation that led Rayleigh first to consider the problem. The reader can verify the general behavior on a sunny day with a sheet of linear polarizer or suitable sunglasses.

(C) Scattering by a small conducting sphere

In this case,

$$\vec{p} = 4\pi\,\epsilon_0\, a^3\, \vec{E}_{inc}$$

$$\vec{m} = -2\pi\, a^3\, \vec{H}_{inc}$$

As before calculate the radiated power and the cross section ...

$$\frac{d\sigma}{d\Omega}[n,\epsilon;n_0,\epsilon_0] = k^4 a^6\, \left|\,\epsilon^{*}\cdot\epsilon_0 - \frac{1}{2}\,(n\times\epsilon^{*})\cdot(n_0\times\epsilon_0)\,\right|^2$$

See Figure 10.3.

fig10p3

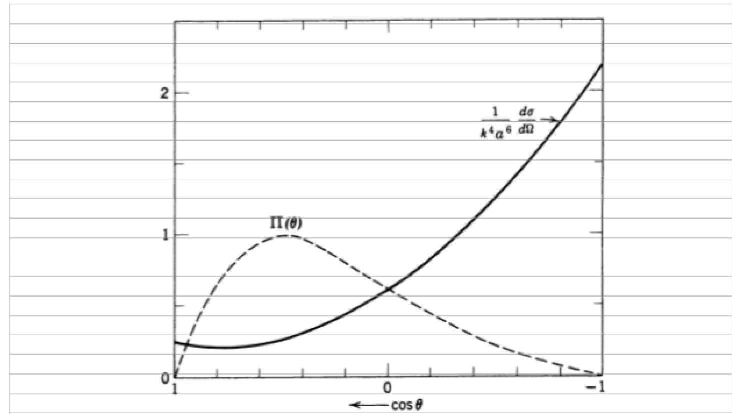


Figure 10.3 Differential scattering cross section (10.16) and polarization of scattered radiation (10.17) for a small perfectly conducting sphere (electric and magnetic dipole approximation).

What can we say about this case?

scan2

The cross section and polarization are plotted versus cos θ in Fig. 10.3. The cross section has a *strong backward peaking* caused by electric dipole–magnetic dipole interference. The polarization reaches $\Pi = +1$ at $\theta = 60°$ and is positive through the whole angular range. The polarization thus tends to be similar to that for a small dielectric sphere, as shown in Fig. 10.2, even though the angular distributions are quite different. The total scattering cross section is $\sigma = 10\pi k^4 a^6/3$, of the same order of magnitude as for the dielectric sphere (10.11) if $(\epsilon_r - 1)$ is not small.

(D) Collection of scatterers

- Interference of waves scattered by different scattering centers;
- $d\sigma/d\Omega = (d\sigma / d\Omega)_0 \times$ form factor;
the form factor depends on the spatial distribution of the scatterers;
- familiar from Bragg scattering — using X-ray scattering to determine crystal structure.