# Scattering and Diffraction

## Section 10.2
## Perturbation Theory of Scattering

Light travels in a straight line in a uniform medium.

What happens in a non–uniform medium?

# (A) General Theory

$$\nabla \cdot \vec{B} = 0 \qquad\qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{D} = 0 \qquad\qquad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \qquad\qquad (21)$$

$$\nabla^2 \vec{D} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{D}}{\partial t^2} = \vec{R} \qquad\qquad (22)$$

$$\vec{R} = -\nabla \times \nabla \times (\vec{D} - \varepsilon_0 \vec{E}) + \varepsilon_0 \frac{\partial}{\partial t} \nabla \times (\vec{B} - \mu_0 \vec{H})$$

Treat the right hand side of (22) as a source, and use the retarded Green's function. Of course the right hand side is not known; but we can use the equation as a starting point for a pertubation expansion.

Also, we'll assume harmonic time dependence; all fields $\propto e^{-i\omega t}$.

$$(\nabla^2 + k^2)\, \vec{D} = \vec{R} \qquad \left(\frac{\partial}{\partial t} = -i\omega\right) \qquad (23)$$

Let $\mathbf{D}^{(0)}$ be the unperturbed solution. Then

$$\vec{D} = \vec{D}^{(0)} + \int d^3x' \frac{e^{ik|\vec{x}-\vec{x}'|}}{4\pi |\vec{x}-\vec{x}'|}\, \vec{R}(\vec{x}) \qquad (24)$$

Far from the scattering region we have an asymptotic equation,

$$\vec{D} \longrightarrow D^{(0)} + \frac{e^{ikr}}{r}\, \vec{A}_{sc} \qquad (25)$$

where $\vec{A}$ (not the vector potential!) is called the *scattering amplitude.*

$$\vec{A}_{sc} = \frac{1}{4\pi} \int d^3x'\, e^{-ik\hat{n}\cdot\vec{x}'}\, \vec{R}(\vec{x}') \qquad (26)$$

With a bit of calculus (leave this as an exercise) this can be rewritten as

$$\vec{A}_{sc} = \frac{k^2}{4\pi} \int d^3x' \, e^{-ik\hat{n}\cdot\vec{x}'} \left\{ \left[ \hat{n} \times (\vec{D} - \epsilon_0 \vec{E}) \right] \times \hat{n} \right. \quad (27)$$
$$\left. - \frac{\epsilon_0 \omega}{k} \hat{n} \times (\vec{B} - \mu_0 \vec{H}) \right\}$$

Looking back at Section 10.1 we can identify the differential cross section as

$$\frac{d\sigma}{d\Omega} = \frac{|\hat{\epsilon}^* \cdot \vec{A}_{sc}|^2}{|\vec{D}^{(0)}|^2} \quad (28)$$

Here $\hat{\epsilon}$ is the polarization of the scattered radiation.

Equations (24), (27), (28) amount to a **formal solution** to the scattering problem.

For Equation (24) we can use the *method of successive approximations.*

$$D^{(0)} \longrightarrow D^{(1)} = D^{(0)} + \delta D^{(1)} \; ;$$

$$D^{(1)} \longrightarrow D^{(2)} = D^{(1)} + \delta D^{(2)} \; ; \; \text{etc}$$

If the scattering is weak then this will converge after only a few iterations.

# (B) The first Born approximation

We' ll consider an example in which $\epsilon$ and $\mu$ vary with position $\vec{x}$. So

$$\vec{D}(\vec{x}) = \left[ \epsilon_0 + \delta\epsilon(\vec{x}) \right] \vec{E}(\vec{x})$$

$$\vec{B}(\vec{x}) = \left[ \mu_0 + \delta\mu(\vec{x}) \right] \vec{H}(\vec{x}) \tag{29}$$

$$\delta\epsilon(\vec{x}) \ll \epsilon_0 \text{ and } \delta\mu(\vec{x}) \ll \mu_0 \, .$$

So, in first-order perturbation theory, approximate the fields on the right-hand side of Eq (xx) by $D^{(0)}$ and $B^{(0)}$

$$\vec{D} - \epsilon_0 \vec{E} = \frac{\delta\epsilon(\vec{x})}{\epsilon_0} \vec{D}^{(0)}(\vec{x})$$

$$\vec{B} - \mu_0 \vec{H} = \frac{\delta\mu(\vec{x})}{\mu_0} \vec{B}^{(0)}(\vec{x}) \tag{30}$$

For a scattering problem, the unperturbed fields are just the incident plane wave

$$\vec{D}^{(0)}(\vec{x}) = \hat{\epsilon}_0 \, D_0 \, e^{ik\hat{n}_0 \cdot \vec{x}}$$

$$\vec{B}^{(0)}(\vec{x}) = \sqrt{\frac{\mu_0}{\epsilon_0}} \; \hat{n}_0 \times \vec{D}^{(0)}(\vec{x})$$

So in this first-order approximation the scattering amplitude becomes

$$\frac{\hat{\epsilon}^* \cdot \vec{A}_{sc}^{(1)}}{D_0} = \frac{k^2}{4\pi} \int dV' \, e^{i\vec{q} \cdot \vec{x}} \left\{ \hat{\epsilon}^* \cdot \hat{\epsilon}_0 \, \frac{\delta\epsilon(\vec{x}')}{\epsilon_0} \right. \tag{31}$$

$$\left. + (\hat{n} \times \hat{\epsilon}^*) \cdot (\hat{n}_0 \times \hat{\epsilon}_0) \, \frac{\delta\mu(\vec{x}')}{\mu_0} \right\}$$

where $\vec{q} = k(\hat{n}_0 - \hat{n})$

<span style="color:red">[like the momentum transfer in quantum mechanics]</span>

Then the cross section is the square of the amplitude.

## Example.
## A long-wavelength approximation

Let's proceed, making the approximation $e^{i\, \vec{q} \cdot \vec{x}} \approx 1$.

I.e., assume the wavelength is large compared to the length scale of the variations $\delta\epsilon$ and $\delta\mu$.

The result is a dipole approximation …

If the variation of the medium consists of a dielectric sphere embedded in the medium, the results are

$$\frac{\hat{\epsilon}^* \cdot A_{SC}}{D_0} = k^2 \frac{\delta\epsilon}{\epsilon_0} (\hat{\epsilon}^* \cdot \hat{\epsilon}_0) \left[ \frac{\sin qa - qa\cos qa}{q^3} \right]$$

$$\lim_{q \to 0} \left( \frac{d\sigma}{d\Omega} \right)_{BORN} = k^4 a^6 \left| \frac{\delta\epsilon(\sigma)}{3\epsilon_0} \right|^2 |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2 \quad (32)$$

In fact this is exactly the same as as the long wavelength result (dipole approximation) in Section 10.1.

## (C) Blue Sky (and the sunset)

Lord Rayleigh (John Strutt);
    published in 1871 and 1899

Friday we discussed the *angular distribution* and *polarization dependence* of the scattering of light by a dipole (Figure 10.2).

Today we'll focus on the
*total cross section.*

For a single molecule, $\vec{p}_i = \epsilon_0\, \gamma_{mol}\, \vec{E}(\vec{x}_i)$ ; recall

$$\delta\epsilon(\vec{x}) = \epsilon_0 \sum_{j} \gamma_{mol}\,(\vec{x} - \vec{x}_j) \qquad (33)$$

Then the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{16\pi^2} |\gamma_{mol}|^2 |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2\, F(\vec{q})$$

The form factor $F(\vec{q}) =$

$$F(\vec{q}) = \sum_{j} \sum_{j'} e^{i\vec{q}\cdot(\vec{x}_j - \vec{x}_{j'})}$$

Incoherent scattering $F = N$

The cross section for randomly
distributed molecules,

$$\sigma \approx \frac{k^4}{6\pi N^2}\left|\epsilon_r - 1\right|^2 = \frac{2k^4}{3\pi N^2}\left|n-1\right|^2 \tag{34}$$

↑ cross section per molecule

Attenuation of the incident beam

$$I(x) = I_0\, e^{-\alpha x}$$

$\alpha$ = attenuation coefficient

$$\alpha = N\sigma \approx \frac{2k^4}{3\pi N}\left|n-1\right|^2 \tag{35}$$
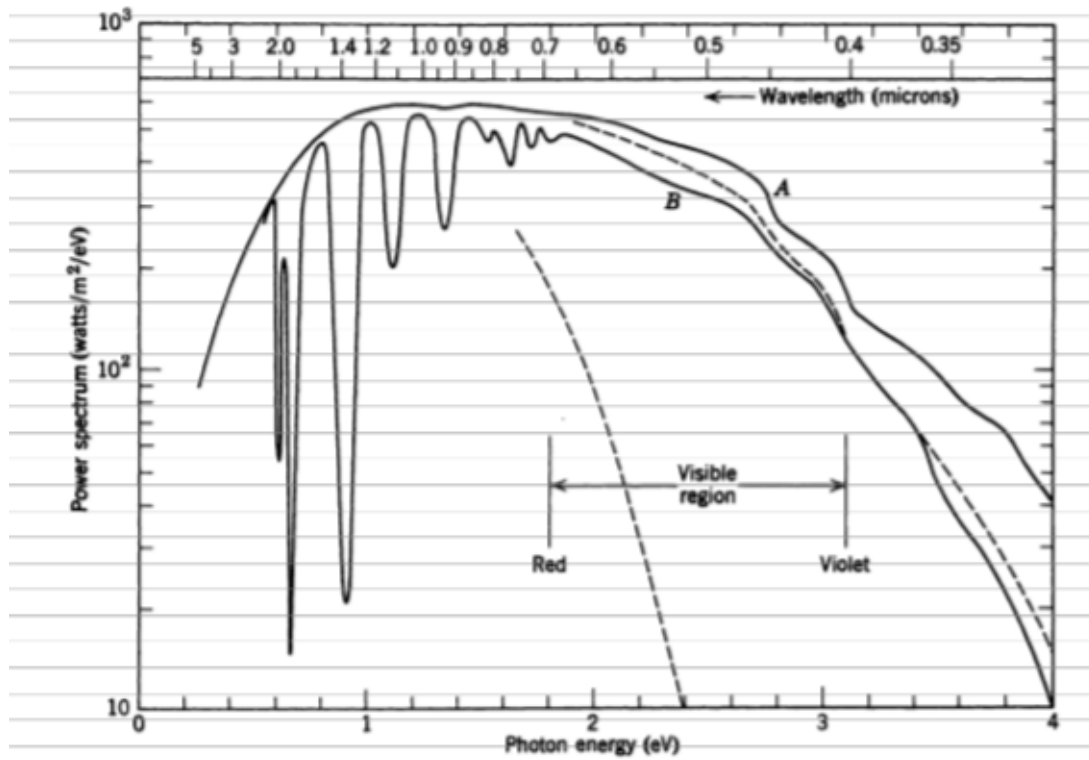
Rayleigh scattering :
the incoherent scattering by randomly
distributed gas molecules.

$$\sigma \propto \omega^4$$

White light from the sun is scattered more for high frequencies (violet and blue) and less for low frequencies (red and orange).

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**Figure 10.4** Power spectrum of solar radiation (in watts per square meter per electron volt) as a function of photon energy (in electron volts). Curve $A$ is the incident spectrum above the atmosphere. Curve $B$ is a typical sea-level spectrum with the sun at the zenith. The absorption bands below 2 eV are chiefly from water vapor and vary from site to site and day to day. The dashed curves give the expected sea-level spectrum at zenith and at sunrise-sunset if the only attenuation is from Rayleigh scattering by a dry, clean atmosphere.

Physically: the spectrum of sky light is peaked at high frequencies;

Biologically: the light of the sky appears blue.

Sunsets are red because at sunset the light travels a long distance and loses its high frequencies by scattering.

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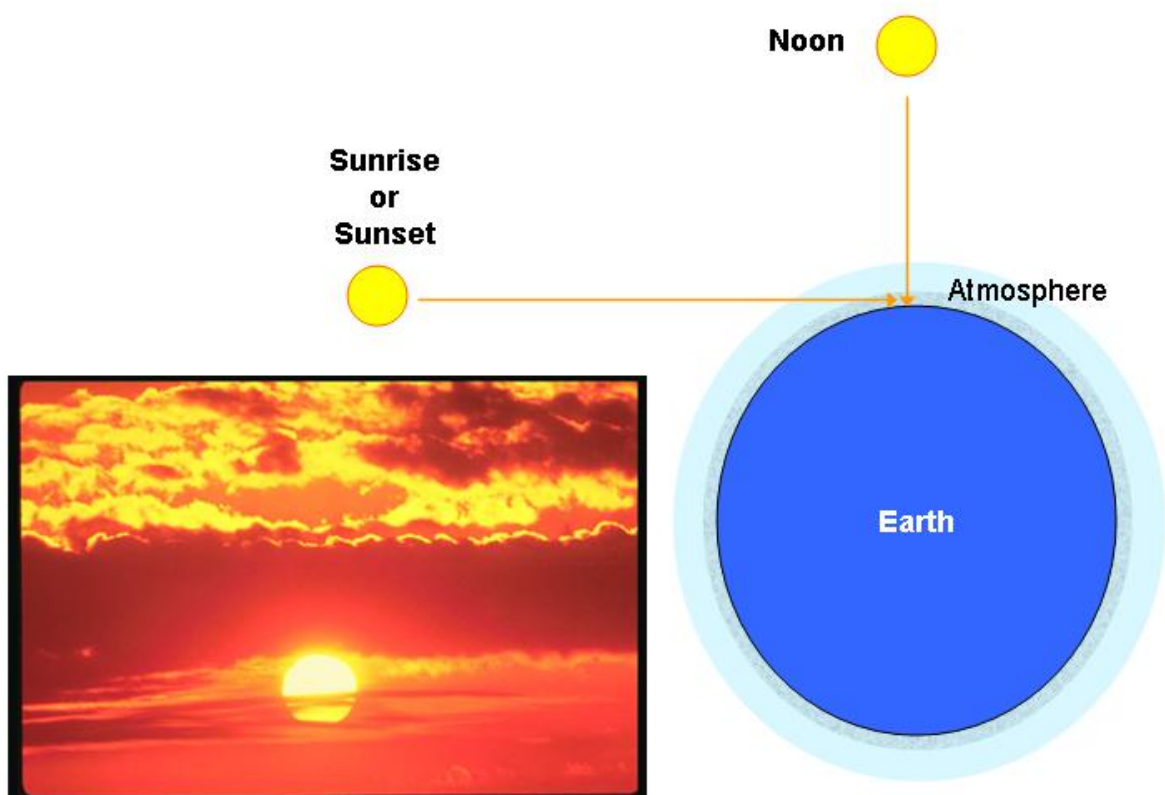
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overhead.* The upper dashed curve is the result expected from curve $A$ if the only attenuation is Rayleigh scattering by a dry, clean, isothermal, exponential atmosphere. In reality the attenuation is greater, mainly because of the presence of water vapor, which has strong absorption bands in the infrared, and ozone, which causes absorption of the ultraviolet, as well as other molecular species and dust. The lower dashed curve indicates roughly the sunrise-sunset spectrum at sea level. Astronauts orbiting the earth see even redder sunsets because the atmospheric path length is doubled.

Detailed observations on the polarization of the scattered light from the sky have been reported.† Just as with the attenuation, the reality departs somewhat from the ideal of a dry, clean atmosphere of low density. At 90° the polarization is a function of wavelength and reaches a maximum of approximately 75% at 5500 Å. It is estimated to be less than 100% because of multiple scattering (6%), molecular anisotropy (6%), ground reflection (5%, and especially important in the green when green vegetation is present), and aerosols (8%).

The formula (10.35) for the extinction coefficient is remarkable in its possession of the factor $N^{-1}$ as well as macroscopic quantities such as the index of refraction. If there were no atomicity ($N \rightarrow \infty$), there would be no attenuation. Conversely, the observed attenuation can be used to determine $N$. This point was urged particularly on Rayleigh by Maxwell in private correspondence. If the properties of the atmosphere are assumed to be well enough known, the relative intensity of the light from a definite star as a function of altitude can be used to determine $N$. Early estimates were made in this way and agree with the results of more conventional methods.