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From the grad student handbook ...

CLASSICAL ELECTRODYNAMICS II (PHY 842)

Electrostatics of conductors
Electrostatics of dielectrics
Microscopic models of dielectric media
Magnetostatics
Para-, dia-, and ferromagnetism
Quasi-stationary fields, skin effect
Electromagnetic waves in material media
propagation, reflection, refraction and polarization
Waveguides and resonant cavities
Scattering and diffraction
Electrodynamics of special media
(plasma, superconductors)
Energy loss by charged particles
Cherenkov radiation.

Topics for the final exam ...

? Electrostatics of dielectrics
? Propagation of plane waves
? Lorentz model of dispersion; $\epsilon(\omega)$
? Waves in a free electron plasma; ω_p
? Skin depth; δ
? Waveguides
? Short answer questions from the homework assignments

Dielectrics

$\nabla\cdot \vec{E} = \rho / \epsilon_0$

$\rho = \rho_{free} - \nabla\cdot \vec{P}$; also could have $\sigma_{bound} = \hat{n} \cdot \vec{P}$

$\nabla\cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_{free}$

$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon \vec{E}$

Maxwell Equations

$\nabla\cdot \vec{B} = 0$ and $\nabla\times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\nabla\cdot \vec{D} = \rho_{free}$ and $\nabla\times \vec{H} = \vec{J}_{free} + \frac{\partial \vec{D}}{\partial t}$

$\vec{D} = \epsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$

Boundary Conditions

$\Delta\ D_{normal} = \sigma_{free}$

$\Delta\ E_{tangential} = 0$

Examples with permittivity

- Parallel plate capacitor : $C = \frac{\epsilon A}{d}$
- Fields for a dielectric sphere in a constant electric field
- Propagation of plane waves : $v_{phase} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n}$
- Reflection and refraction
- Dispersion : $\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \Sigma \frac{\omega_p^2}{\omega_0^2 - \omega^2 + i\omega\gamma}$; $\omega_p^2 = ??$
- Cherenkov radiation

History of Cherenkov Radiation

1888 predicted by Heaviside (but forgotten)

1904 predicted by Sommerfeld (but forgotten)

1910 Marie Curie (noted a blue glow from radium in water)

1926 Lucien Mallet (radium in water)

1934 Pavel Cherenkov (supervisor Sergey Vavilov)

1937 Ilya Frank and Igor Tamm (the theory)

1958 Nobel Prize (Cherenkov, Frank and Tamm)

EQ

The frequency spectrum of Cherenkov radiation by a particle is given by the Frank–Tamm formula:

$$\frac{d^2E}{dx\,d\omega} = \frac{q^2}{4\pi}\mu(\omega)\omega\left(1 - \frac{c^2}{v^2 n^2(\omega)}\right)$$

The Frank–Tamm formula describes the amount of energy E emitted from Cherenkov radiation, per unit length traveled x and per frequency ω. $\mu(\omega)$ is the permeability and $n(\omega)$ is the index of refraction of the material the charge particle moves through. q is the electric charge of the particle, v is the speed of the particle, and c is the speed of light in vacuum.

fig135

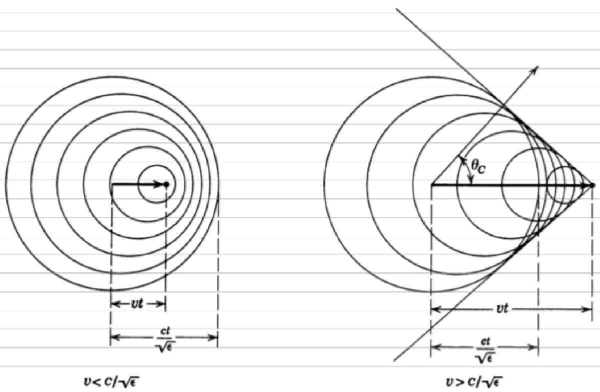


Figure 13.5 Cherenkov radiation. Spherical wavelets of fields of a particle traveling less than and greater than the velocity of light in the medium. For $v > c/\sqrt{\epsilon}$, an electromagnetic "shock" wave appears, moving in the direction given by the Cherenkov angle θ_C.

DERIVATION OF THE FRANK-TAMM FORMULA FOR CHERENKOV RADIATION

§ **Consider** a charged particle moving relativistically along the x-axis, in a medium with index of refraction $n(\omega) = \sqrt{\mu\,\epsilon(\omega)}\,/\sqrt{\mu_0\,\epsilon_0}$. The particle velocity is $\vec{v} = (v,0,0)$ and is approximately constant.

Start with Maxwell's equations; vector and scalar potentials in the Lorenz gauge; and Fourier transform the equations

$(\vec{x},t) \rightarrow (\vec{k},\omega)$. $\Longrightarrow$

$$(k^2 - \mu\epsilon\,\omega^2)\,\Phi = \rho/\epsilon \qquad [\text{functions of } \vec{k},\omega]$$

$$(k^2 - \mu\epsilon\,\omega^2)\,\vec{A} = \mu\,\vec{J}$$

§ **For** a charge of magnitude **ze** (where **e** is the elementary charge) moving with velocity **v**, the charge density and current density can be expressed as

$\rho(\vec{x},t) = ze\,\delta^3(\vec{x} - \vec{v}\,t)$

and $\vec{J}(\vec{x},t) = \vec{v}\,\rho(\vec{x},t)$; taking Fourier transforms,

$$\rho(\vec{k},\omega) = \frac{ze}{2\pi}\,\delta(\omega - \vec{k}\cdot\vec{v})$$

$$\vec{J}(\vec{k},\omega) = \vec{v}\,\rho(\vec{k},\omega)$$

§ **Substituting** these charge and current densities into the wave equation, gives the potentials:

$$\Phi = \frac{ze}{2\pi\epsilon}\,\frac{\delta\left(\omega - \vec{k}\cdot\vec{v}\right)}{k^2 - \mu\,\epsilon\,\omega^2}$$

$$\vec{A} = \mu\epsilon\,\vec{v}\,\Phi$$

§ **These** are the potentials. Now calculate the fields, $\vec{E} = -\nabla\Phi - \partial\vec{A}/\partial t$ and $\vec{B} = \nabla\times\vec{A}$; Fourier transformations...

$$\vec{E} = i\,(\mu\epsilon\,\omega\,\vec{v} - \vec{k}\,)\,\Phi \qquad [\text{functions of } \vec{k},\omega]$$

$$\vec{B} = i\,\mu\epsilon\,\vec{k}\times\vec{v}\,\Phi$$

Check units:

$\mu\epsilon\,\omega v = [sec/m]^2\,[1/sec]\,[m/sec] = [1/m]$;

$\mu\epsilon v E = [sec/m]^2\,[m/sec]\,\,[V/m] = [Vs/m^2]$

§ **Field components**

Consider the electric field as a function of frequency at a point at some perpendicular distance b from the particle trajectory;

i.e., $\vec{E}(\vec{x},t)$ at $\vec{x} = (\,0,\,b,\,0\,)$.

In frequency space,

$$\vec{E}(\vec{x},\omega) = \int \frac{d^3k}{(2\pi)^{3/2}}\,\vec{E}(\vec{k},\omega)\,e^{ibk_y}$$

§ **First** we compute x component of $\vec{E}$.

The integrand is $\quad i(\mu\epsilon\omega\, v - k_x)\,\Phi$.

The integrals over k_x and k_z are straightforward, and that leaves $[\vec{x} = (0b0)]$

$$E_x(\omega) = \frac{-ize\omega\pi}{(2\pi)^{5/2}}\,\frac{\kappa^2}{\omega^2}\int_{-\infty}^{\infty} dk_y\,\frac{e^{ibk_y}}{[\,k_y^2 + \kappa^2]^{1/2}}$$

where $\kappa^2 = \omega^2\,(\,1/v^2 - \mu\epsilon)$

Use Mathematica to calculate the integral,

```
(* Using Mathematica *)
Integrate[Cos[17 * x] / Sqrt[1 + x^2], {x, -Infinity, Infinity}]
```

```
2 BesselK[0, 17]
```

$$E_x(\omega) = \frac{-ize}{(2\pi)^{3/2}\,\epsilon}\,\frac{\kappa^2}{\omega}\,K_0(\kappa b)$$

§ Similarly,

$$E_y(\omega) = \frac{ze}{(2\pi)^{3/2}\epsilon}\,\frac{\kappa}{v}\,K_1(\kappa b)$$

$$E_z = 0$$

$$B_x = B_y = 0$$

$$B_z(\omega) = \mu\epsilon\, v\, E_y(\omega)$$

§ Radiated energy

■ Let δU = the radiated energy when the particle traverses distance δx_p .

■ Let P_a = the power passing through a cylinder of radius a around the x axis. (a is the same as b)

By energy conservation,

$$\left(\frac{\delta U}{\delta x_p}\right)_{rad} = \frac{P_a}{v} = \frac{1}{v}\int_{-\infty}^{\infty} \hat{\rho}\cdot(\vec{E}\times\vec{H})\ d(\text{area})$$

$$= \frac{1}{v\mu}\int_{-\infty}^{\infty} (-E_x B_z)\ dx\ \ 2\pi\alpha$$

Change the integration: The integral over dx at one instant of time is equal to the integral over time at any one position; ∴ change dx = v dt ⟹

$$\left(\frac{\delta U}{\delta x_p}\right)_{rad} = \frac{-2\,\pi\,a}{\mu} \int_{-\infty}^{\infty} E_x(t)\, B_z(t)\, dt$$

Convert it to a frequency integral and use complex fields ⟹

$$\left(\frac{\delta U}{\delta x_p}\right)_{rad} = \frac{-4\,\pi\,a}{\mu}\, \mathrm{Re}\left\{ \int_0^{\infty} E_x(\omega)\, B_z^{\star}(\omega)\, d\omega \right\}$$

§ We can make an approximation

For macroscopic Cherenkov radiation, consider b ≫ atomic dimensions and b ≫ wavelengths of the Cherenkov light; then

$$\kappa b \propto \frac{\omega\, b}{c} = \frac{2\,\pi\, b}{\text{wavelength}} \gg 1$$

$$K_m[\kappa b] \sim \sqrt{\frac{\pi}{2\,\kappa b}}\; e^{-\kappa b}$$

```
(* Verify using Mathematica *)
Series[BesselK[0, x], {x, Infinity, 1}]
Series[BesselK[1, x], {x, Infinity, 1}]
```

$$e^{-x+O\left[\frac{1}{x}\right]^2} \left(\sqrt{\frac{\pi}{2}}\sqrt{\frac{1}{x}} + O\left[\frac{1}{x}\right]^{3/2}\right)$$

$$e^{-x+O\left[\frac{1}{x}\right]^2} \left(\sqrt{\frac{\pi}{2}}\sqrt{\frac{1}{x}} + O\left[\frac{1}{x}\right]^{3/2}\right)$$

Thus,

$$E_x(\omega) \longrightarrow \frac{-ize}{(2\pi)^{3/2}\,\epsilon}\;\frac{\kappa^2}{\omega}\;\sqrt{\frac{\pi}{2\,\kappa\,a}}\;e^{-\kappa a})$$

$$= \frac{-ize}{4\,\pi\epsilon}\;\frac{\kappa^2}{\omega}\;\frac{e^{-\kappa a}}{\sqrt{\kappa a}}$$

Similarly,

```
Style["Ey(ω) → ze/(4 πϵ) κ/v e^-κa/√(κa)
AND   Bz(ω) = μϵ v Ey(ω)", ff]
```

$$E_y(\omega) \longrightarrow \frac{ze}{4\,\pi\epsilon}\;\frac{\kappa}{v}\;\frac{e^{-\kappa a}}{\sqrt{\kappa a}}$$

$$\text{AND}\quad B_z(\omega) = \mu\epsilon\, v\, E_y(\omega)$$

$$\left(\frac{\delta U}{\delta x_p}\right)_{rad} = \frac{-4\,\pi\,a}{\mu}\,\mathrm{Re}\left\{\int_0^\infty E_x(\omega)\,B_z^\star(\omega)\,d\omega\right\}$$

The rest is just algebra with complex numbers (about 2 pages long) $\Longrightarrow$

$$\left(\frac{\delta U}{\delta x_p}\right)_{rad} = \int_0^\infty d\omega\,\mathrm{Re}\{-i\sqrt{\kappa^\star/\kappa}\}\;\frac{\mu}{4\,\pi}\;z^2 e^2\;\cdot\;\omega\left(1-\frac{1}{\beta^2\,\mu_r\,\epsilon_r}\right)e^{-(\kappa+\kappa^\star)\,a}$$

WHERE $\beta = v/c$

If κ is purely imaginary then the formula collapses to

$$\left(\frac{\delta U}{\delta x_p}\right)_{rad} = \int_0^\infty d\omega\;\frac{\mu}{4\,\pi}\;z^2 e^2\;\omega\left(1-\frac{\mu_0\,\epsilon_0}{\beta^2\,\mu\epsilon\,(\omega)}\right)$$

independent of the radius a ;

this is the Cherenkov radiation.

When is κ purely imaginary?

We defined ...

$$\kappa^2 = \omega^2 \left(1/v^2 - \mu\epsilon \right)$$

OR, $$\kappa^2 = \omega^2 \left(\frac{1}{v^2} - \frac{1}{v_{phase}^2} \right)$$

κ is purely imaginary if $v > v_{phase}$;

i.e., κ is purely imaginary if the particle is moving faster than the speed of light in the medium.

The energy radiated by the moving charge

$$\left(\frac{\delta U}{\delta x_p} \right)_{rad} = \frac{\mu z^2 e^2}{4\pi} \int_{v > c/n(\omega)} \omega \left(1 - \frac{c^2}{v^2 n^2(\omega)}\right) d\omega$$

This is the Frank–Tamm equation.

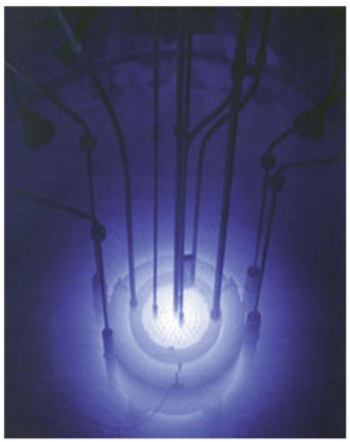
References: Jackson, Section xxx

Wikipedia, " Frank Tamm formula"

(probably written by Jackson, but in Gaussian units).

Example: water for visible light has $n(\omega) \approx 1.33$.
Why is Cherenkov light blue?
What is the minimum particle speed to produce Cherenkov radiation in water?

fig1



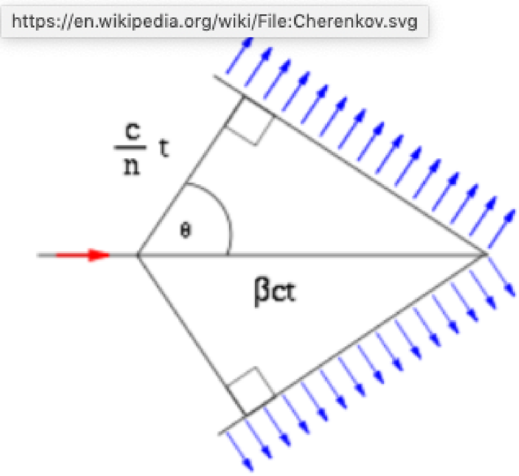
Cherenkov radiation in the Reed Research Reactor.

Cherenkov light forms a cone around the trajectory of the charged particle.
Calculate the angle of the cone.

Answer: $\cos\theta = \frac{1}{n\beta}$

Note $\beta > \frac{1}{n}$ because $v > \frac{c}{n}$ = speed of light.

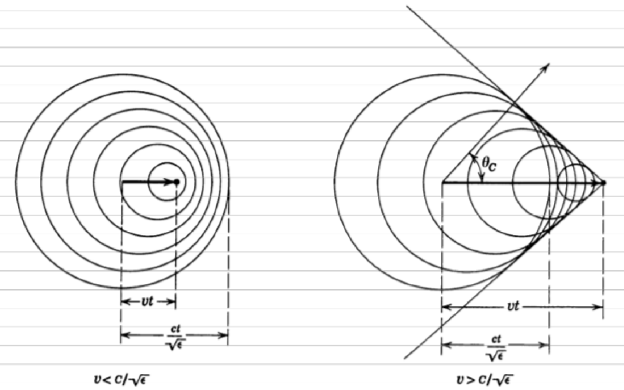
fig2



The geometry of the Cherenkov radiation shown for the ideal case of no dispersion.

Illustration as a shockwave

fig3



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