ELECTROMAGNETIC WAVES

in various contexts

- free space; c = $\frac{1}{\sqrt{\mu_0 \epsilon_0}}$
- simple linear materials

  dielectrics

  $v_{\text{phase}} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n} \Longrightarrow$ optics
- conductors and plasmas
- waveguides

  What is the speed of EM waves in a wave guide?

Maxwell Equations

where there are no free charges or currents

$\nabla \cdot \vec{B} = 0$	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
$\nabla \cdot \vec{D} = 0$	$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$

For simple materials,

$\vec{D}(\vec{x},t) = \epsilon \vec{E}(\vec{x},t)$, constant ϵ

$\vec{B}(\vec{x},t) = \mu \vec{H}(\vec{x},t)$, constant μ

In free space, $\mu = \mu_0$ and $\epsilon = \epsilon_0$.

The PDEs

$\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ obey the wave equation.

sc1

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$$
$$= -\nabla^2 \vec{E}$$
$$\hookrightarrow = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\mu \frac{\partial}{\partial t}(\nabla \times \vec{H})$$
$$= -\mu \frac{\partial}{\partial t} \epsilon \frac{\partial \vec{E}}{\partial t} = -\mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$
$$\therefore \nabla^2 \vec{E} - \mu\epsilon \frac{\partial^2}{\partial t^2} \vec{E} = 0.$$

Plane wave solutions in a uniform medium (including free space); and so far we are not specifying any boundaries

$$\vec{E} = \vec{E}_0 \cos(\vec{k} \cdot \vec{x} - \omega t)$$

$\vec{E} = \vec{E}_0 \cos(\vec{k} \cdot \vec{x} - \omega t)$	
$\nabla \cdot \vec{D} = 0$	$\vec{k} \cdot \vec{E}_0 = 0$
$(\nabla^2 - \mu\epsilon \frac{\partial^2}{\partial t^2}) \vec{E} = 0$	$-k^2 + \mu\epsilon\, \omega^2 = 0$
$\frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} = v_{phase} = \frac{c}{n}$	

But there is more to it …

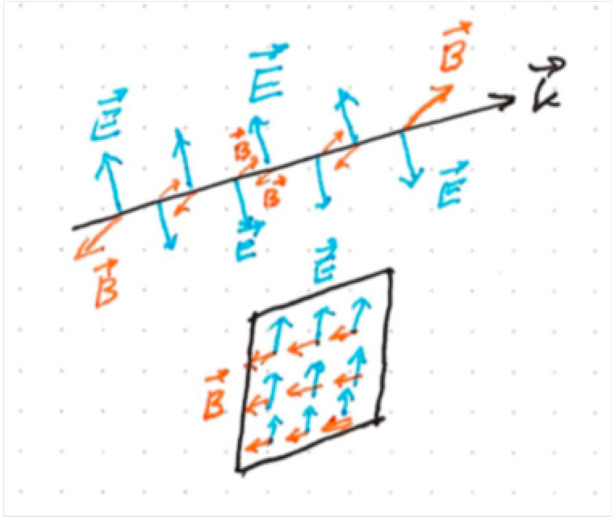
$$\frac{\partial \vec{B}}{\partial t} = -\nabla\times\vec{E} = \vec{k}\times\vec{E}_0\ \sin(\vec{k}\bullet\vec{x} - \omega t)$$

$$\therefore \vec{B} = \frac{\vec{k}\times\vec{E}_0}{\omega}\cos(\vec{k}\bullet\vec{x} - \omega t)$$

$$= \vec{B}_0\cos(\vec{k}\bullet\vec{x} - \omega t)$$

$$\vec{B}_0 = \frac{\vec{k}\times\vec{E}_0}{\omega} = \frac{1}{v_{\text{phase}}}\hat{k}\times\vec{E}_0$$

sc2



The wave fronts are planes $\perp$ to $\vec{k}$.

EM waves in an enclosed space — waveguides

The PDFs are the same, but now we have boundary surfaces!

A rectangular waveguide, with dimensions $\delta x = a$ and $\delta y = a/2$; waves propagating in the z direction.

Inside is free space; $\mu = \mu_o$ and $\epsilon = \epsilon_o$. The walls are metal, which we'll approximate by a perfect conductor.

So on the surfaces,

$$\vec{E}_{\text{tangential}} = 0 \quad \text{and} \quad B_{\text{normal}} = 0\,;$$

$$E_{\text{normal}} = \Sigma / \epsilon_0 \quad \text{and} \quad \vec{B}_{\text{tangential}} = \mu_0\, \hat{n} \times \vec{K}\,.$$

We are looking for solutions such that the fields are $\propto e^{i(kz-\omega t)}$;

(using complex functions; the Re Part is understood) ;

so $\frac{\partial}{\partial z} \longrightarrow ik$ and $\frac{\partial}{\partial t} \longrightarrow -i\omega$.

Out[•]=

$\vec{F} = \hat{z}\,F_z + \vec{F}_T$ where $\vec{F}_T = \hat{x}\,F_x + \hat{y}\,F_y$	
$\nabla \bullet \vec{D} = 0$	$ik\,E_z + \nabla_T \bullet \vec{E}_T = 0$
$\nabla \bullet \vec{B} = 0$	$ik\,B_z + \nabla_T \bullet \vec{B}_T = 0$
$i\omega\,\vec{B} = \nabla \times \vec{E}$	$i\omega\mu\,\vec{H} = (\hat{z}ik + \nabla_T) \times (\hat{z}E_z + \vec{E}_T)$
$-i\omega\,\vec{D} = \nabla \times \vec{H}$	$-i\omega\epsilon\,\vec{E} = (\hat{z}ik + \nabla_T) \times (\hat{z}H_z + \vec{H}_T)$

Separate the curl equations into longitudinal and transverse components,

$$i\omega\mu\, H_z = \hat{z} \bullet (\nabla_T \times \vec{E}_T)$$

$$i\omega\mu\, \vec{H}_T = \hat{z} \times (\, ik\, \vec{E}_T - \nabla_T E_z\,)$$

and

$$-i\omega\epsilon\, E_z = \hat{z} \bullet (\nabla_T \times \vec{H}_T)$$

$$-i\omega\epsilon\, \vec{E}_T = \hat{z} \times (\, ik\, \vec{H}_T - \nabla_T H_z\,)$$

And now solve for $\vec{E}_T$ and $\vec{H}_T$ in terms of E_z and H_z ...

$$i\omega\mu\, \vec{H}_T = ik\, \hat{z}\times \vec{E}_T - \hat{z}\times \nabla_T E_z$$

$$= ik\, \hat{z} \times [\, \frac{i}{\omega\epsilon}\, \hat{z} \times (ik\, \vec{H}_T - \nabla_T\, H_z)] - \hat{z}\times \nabla_T E_z$$

$$(i\omega\mu - \frac{ik^2}{\omega\epsilon}\}\, \vec{H}_T = -\frac{k}{\omega\epsilon}\, \nabla_T H_z - \hat{z}\times \nabla_T E_z$$

$$\vec{H}_T = \frac{ik}{(\mu\epsilon\omega^2 - k^2)}\, [\, \nabla_T H_z - \sqrt{\epsilon/\mu}\, \hat{z}\times \nabla_T E_z\,]$$

and similarly

$$\vec{E}_T = \frac{ik}{(\mu\epsilon\omega^2 - k^2)}\, [\, \nabla_T E_z - \sqrt{\mu/\epsilon}\, \hat{z}\times \nabla_T H_z\,]$$

Let $\gamma^2 = \mu\epsilon\omega^2 - k^2$

TE waves

$E_z = 0$ and $H_z = H_0\,\psi(\mathrm{x,y})\; e^{i(\mathrm{kz}-\omega \mathrm{t})}$

$E_x = \frac{\mathrm{ik}}{\gamma^2}\sqrt{\frac{\mu}{\epsilon}}\,\frac{\partial H_z}{\partial y}$ and $E_y = \frac{-\mathrm{ik}}{\gamma^2}\sqrt{\frac{\mu}{\epsilon}}\,\frac{\partial H_z}{\partial x}$

The wave equation $\Longrightarrow$

$$(\nabla^2 - \mu\epsilon\,\partial_{\mathrm{t}}^2)\,\mathrm{H_z} = 0$$

$$(\nabla_{\mathrm{T}}^2 + \mu\epsilon\omega^2 - \mathrm{k}^2)\,\psi = 0$$

–– with boundary conditions ––

$$\partial_{\mathrm{x}}\psi(0,\mathrm{y}) = \partial_{\mathrm{x}}\psi(\mathrm{a,y}) = \partial_{\mathrm{y}}\psi(\mathrm{x},0) = \partial_{\mathrm{y}}\psi(\mathrm{x,a}/2) = 0$$

–– implies ––

$\psi(\mathrm{x,y}) = \cos(\frac{\mathrm{m}\pi \mathrm{x}}{\mathrm{a}})\,\cos(\frac{\mathrm{n}\pi \mathrm{y}}{\mathrm{a}/2})$ where $\mathrm{m,n} \in \mathbb{Z}$

and $\quad \mu\epsilon\omega^2 - \mathrm{k}^2 = (\frac{\mathrm{m}\pi}{\mathrm{a}})^2 + (\frac{\mathrm{n}\pi}{\mathrm{a}/2})^2 \equiv \gamma^2$

★ The cutoff frequency ω_{mn}

Propagation $\propto e^{i(\mathrm{kz}-\omega \mathrm{t})}$ requires that k is real; i.e., $k^2 > 0$.
Then we must have $\omega > \omega_{\mathrm{mn}}$ where

$$\mu\epsilon\,\omega_{\mathrm{mn}}^2 = \gamma_{\mathrm{mn}}^2 = (\frac{\mathrm{m}\pi}{\mathrm{a}})^2 + (\frac{\mathrm{n}\pi}{\mathrm{a}/2})^2$$

Note: In the limit $\mathrm{a} \to \infty$, $\omega_{\mathrm{mn}} \to 0$;
the frequency cutoff goes away.

★ The speed of EM waves in the waveguide

$$v_{phase} = \frac{\omega}{k} = \frac{c\omega}{\sqrt{\omega^2 - \omega_{mn}^2}} > c$$

$$v_{group} = \frac{d\omega}{dk} = \frac{c\sqrt{\omega^2 - \omega_{mn}^2}}{\omega} < c$$

Note: In the limit $a \to \infty$,
v_{phase} and $v_{group} \to c$, as we know.

★ Surface current densities

Consider the case {mn} = {10}.

$\psi(x,y) = \cos(\pi x/a)$ AND $\gamma = \frac{\pi}{a}$

$$\vec{H}_T = \frac{ikH_0}{\gamma^2}\,\hat{x}\left(\frac{-\pi}{a}\right)\sin(\pi x/a)\,e^{i(kz-\omega t)}$$

$$\vec{H}_T = \frac{ka\,H_0}{\pi}\,\hat{x}\,\sin(\pi x/a)\,\sin(kz-\omega t)$$

$$\vec{K} = \hat{n}\times\vec{H} = H_0\hat{n}\times[$$
$$\hat{x}\,\frac{ka}{\pi}\,\sin(\pi x/a)\sin(kz-\omega t)$$
$$+\,\hat{z}\,\cos(\pi x/a)\cos(kz-\omega t)\quad]$$

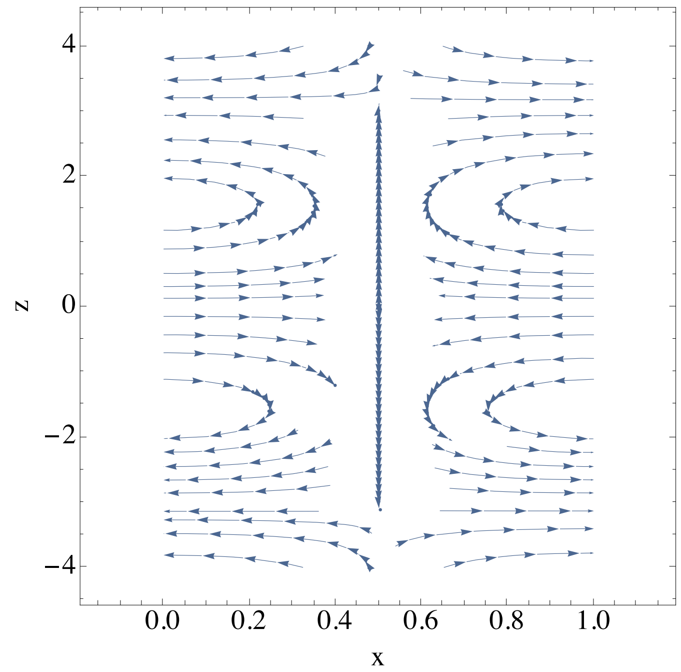
- The walls at x = 0 and x = a
have $\hat{n} = \pm\hat{x}$; there

$$\vec{K} = \pm\,\hat{y}\,H_0\cos(\mathrm{kz}\text{-}\omega\mathrm{t}).$$

- The walls at y = 0 and y = a/2
have $\hat{n} = \pm\hat{y}$; there

$$\vec{K} = \pm\,\hat{z}\,H_0\,\frac{\mathrm{ka}}{\pi}\,\sin(\pi\mathrm{x/a})\,\sin(\mathrm{kz}\text{-}\omega\mathrm{t})$$
$$\mp\,\hat{x}\,H_0\cos(\pi\mathrm{x/a})\,\cos(\mathrm{kz}\text{-}\omega\mathrm{t})$$


```
a = 1; k = 1 / a;
fz[x_, z_] = Sin[Pi * x / a] * Sin[k * z];
fx[x_, z_] = Cos[Pi * x / a] * Cos[k * z];
StreamPlot[{fx[x, z], fz[x, z]},
 {x, 0, 1}, {z, -4, 4}, FrameLabel → {"x", "z"},
 BaseStyle → {ff, 24}, ImageSize → Large]
```



Waves in a conducting medium

Recall homework problem 8-3.

8-3. (a) Derive the wave equation for an electromagnetic wave in a material with permittivity ϵ and conductivity g. (Ignore magnetization.)
(b) Solve the equation for a plane wave with frequency ω.
(c) Calculate the absorption length for a good conductor.
(What is meant by a "good" conductor?)

(A) $\nabla\times\vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$ and $\nabla\times\vec{H} = g\vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t}$.

Evaluate $\nabla\times(\nabla\times\vec{E})$ and simplify

$\Rightarrow \mu_0 \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \nabla^2\vec{E} + \mu_0 g \frac{\partial \vec{E}}{\partial t} = 0$

(B) A plane wave solution has $\vec{E} \propto e^{i(kz-\omega t)}$.
Solution requires $k^2 = \mu_0\epsilon\omega^2 + i\,\mu_0 g\omega$.

(C) Write $k = \beta + \frac{i\alpha}{2}$ where α is the absorption coefficient.

Absorption length = $\delta = 1/\alpha$.

For a good conductor, $\delta \approx \sqrt{\frac{2}{\mu_0 g\omega}}$

(same as the "skin depth" defined in Section 5.18).