Physics 472 - 2020

Quantum Mechanics

Problem Set 5

1. Consider two particles, with masses m_1, m_2, coordinates $\mathbf{r}_1, \mathbf{r}_2$, and momenta $\mathbf{p}_1, \mathbf{p}_2$, which interact with each other through the central potential $U(|\mathbf{r}_1 - \mathbf{r}_2|)$. Make a transition to new variables: the coordinate and the momentum of the center of mass
$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M}, \quad \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 \quad (M = m_1 + m_2),$$
and the coordinate and the momentum of relative motion
$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \quad \mathbf{p} = \frac{m_2\mathbf{p}_1 - m_1\mathbf{p}_2}{M}$$
Find the commutation relations for the components of these new coordinates and momenta,
$$[r_\alpha, p_\beta],\ [R_\alpha, P_\beta],\ [r_\alpha, P_\beta],\ [p_\alpha, R_\beta]$$
where $\alpha, \beta = 1, 2, 3$ enumerate the projections on x, y, z axes.

2. Show that the total Hamiltonian in the previous problem can be written as
$$H = \frac{\mathbf{P}^2}{2M} + \frac{\mathbf{p}^2}{2\mu} + U(r)$$
with $\mu = m_1 m_2/(m_1 + m_2)$.

3. Apply the Hamiltonian of the previous problem (without derivation) to a hydrogen atom, assuming that particle 1 is a proton and particle 2 is an electron. Write the wave function of the hydrogen atom in the ground state of the relative motion of the electron and proton, taking into account the translational motion of the atom. Use variables $\mathbf{R}$ and $\mathbf{r}$. Find the energy of the ground state of the hydrogen atom assuming that the atom as a whole is at rest and taking into account the finite proton mass.

4. Find the ground-state energy of a positronium at rest (the binding energy). Positronium is something you can be asked about on a GRE test. It is an analog of the hydrogen atom, with the proton replaced by a positron.

Each problem is 10 pt.