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Problem Set 3

1. The explicit expression for Y_ℓ^m is

$$Y_\ell^m(\theta,\varphi) = C_\ell^m e^{im\varphi} P_\ell^m(\cos\theta),$$

where $P_\ell^m(x) = (1-x^2)^{|m|/2}\left(\frac{d}{dx}\right)^{|m|} P_\ell(x)$

and the Legendre polynomial is written as

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell.$$

We showed in class that, for the parity operator $\hat{P}$, we have $P f(\theta,\varphi) = f(\pi-\theta, \varphi+\pi)$. Respectively

$$P Y_\ell^m(\theta,\varphi) = C_\ell^m e^{im(\varphi+\pi)} P_\ell^m(-\cos\theta).$$

From the definition, $P_\ell(-x) = \frac{1}{2^\ell \ell!}\left(-\frac{d}{dx}\right)^\ell (x^2-1)^\ell = (-1)^\ell P_\ell(x)$

$$P_\ell^m(-x) = (1-x^2)^{|m|/2}\left(-\frac{d}{dx}\right)^{|m|} P_\ell(-x) = (-1)^{|m|}(-1)^\ell P_\ell^m(x)$$

$$e^{im(\varphi+\pi)} = e^{im\pi} e^{im\varphi} = (-1)^{|m|} e^{im\varphi}.$$

$$Y_\ell^m(\pi-\theta, \varphi+\pi) = (-1)^{|m|}(-1)^{|m|}(-1)^\ell Y_\ell^m(\theta,\varphi) = (-1)^\ell Y_\ell^m(\theta,\varphi)$$

$\Rightarrow$ as expected

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2. We have $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Then $\sigma_x \sigma_y = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_z$; $\sigma_y \sigma_z = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma_x$,

$\sigma_z \sigma_x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_y$.

We have $\sigma_y \sigma_x = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma_z$ and similarly

$\sigma_z \sigma_y = -i\sigma_x$, $\sigma_x \sigma_z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_y$. Therefore

$[S_x, S_y] = \frac{\hbar^2}{4}[\sigma_x, \sigma_y] = 2i\frac{\hbar^2}{4}\sigma_z = i\hbar\left(\frac{\hbar}{2}\sigma_z\right) = i\hbar S_z$

Similarly

$[S_y, S_z] = \frac{\hbar^2}{4}[\sigma_y, \sigma_z] = 2i\frac{\hbar^2}{4}\sigma_x = i\hbar S_x$

$[S_z, S_x] = \frac{\hbar^2}{4}[\sigma_z, \sigma_x] = 2i\frac{\hbar^2}{4}\sigma_y = i\hbar S_y$

The relation $[S^2, S_x] = [S^2, S_y] = [S^2, S_z]$ follows from the above relations, as for the orbital angular momentum. For example,

$[S^2, S_x] = [S_x^2 + S_y^2 + S_z^2, S_x] = S_y[S_y, S_x] + [S_y, S_x]S_y$

$+ S_z[S_z, S_x] + [S_z, S_x]S_z =$

$= -i\hbar S_y S_z - i\hbar S_z S_y + i\hbar S_z S_y + i\hbar S_y S_z = 0$

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3. For $s=1$ we have $m_s=-1,0,1$. Therefore the eigenfunctions of S_z in matrix notations, are $\chi_{-1}=\begin{pmatrix}0\\0\\1\end{pmatrix}$, $\chi_0=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\chi_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$, and $S_z\chi_\alpha=\hbar\alpha\chi_\alpha$, $\alpha=-1,0,1$. These eigenfunctions are orthogonal and normalized. For example $\langle\chi_{-1}|\chi_1\rangle=(0\,0\,1)\begin{pmatrix}1\\0\\0\end{pmatrix}=0$, $\langle\chi_0|\chi_0\rangle=(0\,1\,0)\begin{pmatrix}0\\1\\0\end{pmatrix}=1$. Since S_z has such simple eigenvalues, we have $S_z=\hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$ (S_z is a diagonal matrix in the chosen basis)

We have $S_+\chi_{-1}=\hbar\sqrt{s(s+1)-m(m+1)}\Big|_{s=1,m=-1}\chi_0=\hbar\sqrt{2}\,\chi_0$

$$S_+\chi_0=\hbar\sqrt{s(s+1)-m(m+1)}\Big|_{s=1,m=0}\chi_1=\hbar\sqrt{2}\,\chi_1$$

$$S_+\chi_1=0$$

$S_+=\hbar\begin{pmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{pmatrix}$; $S_+\chi_{-1}=\begin{pmatrix}a_3\\b_3\\c_3\end{pmatrix}=\sqrt{2}\begin{pmatrix}0\\1\\0\end{pmatrix}$; $a_3=c_3=0$, $b_3=\sqrt{2}$

$S_+\chi_0=\begin{pmatrix}a_2\\b_2\\c_2\end{pmatrix}=\sqrt{2}\begin{pmatrix}1\\0\\0\end{pmatrix}$; $a_2=\sqrt{2}$, $b_2=c_2=0$

$S_+\chi_1=\begin{pmatrix}a_1\\b_1\\c_1\end{pmatrix}=0$; $a_1=b_1=c_1=0$; $S_+=\hbar\sqrt{2}\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$

$S_-=\hbar\sqrt{2}\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}$; $S_x=\frac{1}{2}(S_++S_-)=\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}$,

$S_y=\frac{-i}{2}(S_+-S_-)=\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}$. Check $S_xS_y-S_yS_x$

$$=\frac{\hbar^2}{2}\left[\begin{pmatrix}i&0&-i\\0&0&0\\i&0&-i\end{pmatrix}-\begin{pmatrix}-i&0&-i\\0&0&0\\i&0&i\end{pmatrix}=\begin{pmatrix}2i&0&0\\0&0&0\\0&0&-2i\end{pmatrix}\right]=i\hbar S_z$$

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4. We have $\langle \chi|\chi\rangle = c^2(a^* \; b^*)\begin{pmatrix} a \\ b \end{pmatrix} = c^2(|a|^2+|b|^2) = 1$. Hence $c = \frac{1}{\sqrt{|a|^2+|b|^2}}$

the normalized spinor is $\chi = \frac{1}{\sqrt{|a|^2+|b|^2}}\begin{pmatrix} a \\ b \end{pmatrix}$

$$\langle \chi|S_x|\chi\rangle = \frac{\hbar}{2(|a|^2+|b|^2)}(a^* \; b^*)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2(|a|^2+|b|^2)}(a^* \; b^*)\begin{pmatrix} b \\ a \end{pmatrix} = \frac{\hbar \, Re(ab^*)}{|a|^2+|b|^2}$$

$$\langle \chi|S_y|\chi) = \frac{\hbar}{2(|a|^2+|b|^2)}(a^* \; b^*)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2(|a|^2+|b|^2)}(a^* \; b^*)\begin{pmatrix} -ib \\ ia \end{pmatrix} = \frac{\hbar \, Im(a^*b)}{|a|^2+|b|^2}$$

$$\langle \chi|S_z|\chi\rangle = \frac{\hbar}{2(|a|^2+|b|^2)}(a^* \; b^*)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2(|a|^2+|b|^2)}(a^* \; b^*)\begin{pmatrix} a \\ -b \end{pmatrix} = \frac{\hbar(|a|^2-|b|^2)}{2(|a|^2+|b|^2)}$$

Part 2. Use $\sigma_+ = \sigma_x + i\sigma_y = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, $\sigma_- = (\sigma_x - i\sigma_y) = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$

$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \frac{1}{2}\alpha(\sigma_z + I) + \frac{1}{2}\beta\sigma_+ + \frac{1}{2}\gamma\sigma_- - \frac{1}{2}\delta(\sigma_z - I)$$

$$= \frac{1}{2}(\alpha-\delta)\sigma_z + \frac{1}{2}(\alpha+\delta)I + \frac{1}{2}(\beta+\gamma)\sigma_x + \frac{i}{2}(\beta-\gamma)\sigma_y$$