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Problem Set 4

1. The classical equation of motion for the angular momentum $\vec{M}$ is

$$\frac{d\vec{M}}{dt} = \text{Torque} = \vec{\mu} \times \vec{B}$$

Applying this to spin. In the operator form, $\vec{M} \Rightarrow \vec{S}$, $\vec{\mu} \Rightarrow -\frac{g\mu_B}{\hbar}\vec{S}$. With $g=2$ we have

$$\frac{d\vec{S}}{dt} = -\omega_L \, \vec{S} \times \frac{\vec{B}}{B}, \quad \omega_L = \frac{|e|B}{m},$$

it is called the cyclotron frequency, but the in this case also the Larmor frequency "consistent" name is

In components, for the expectation values (since $\vec{B} = B\hat{z}$)

$$\frac{d\langle S_x\rangle}{dt} = -\omega_L \langle S_y\rangle$$

$$\frac{d\langle S_y\rangle}{dt} = \omega_L \langle S_x\rangle$$

The initial conditions: $\langle S_x(t)\rangle_{t=0}$ and $\langle S_y(t)\rangle_{t=0}$ can be written as $\langle S_x(0)\rangle = A\cos\varphi$, $\langle S_y(0)\rangle = A\sin\varphi$; these expressions just define A and φ. Then the solution of the equations of motion is

$$\langle S_x(t)\rangle = A\cos(\omega_L t + \varphi), \quad \langle S_y(t)\rangle = A\sin(\omega_L t + \varphi)$$

The frequency ω_L coincides with the Zeeman level splitting found in class (divided by $\hbar$)

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2. The Hamiltonian is $H = -\vec{\mu}\cdot\vec{B}$, with the magnetic moment $\vec{\mu} = -\frac{1}{\hbar}\mu_B \vec{L} - \frac{1}{\hbar}\mu_B g \vec{S}$, $\mu_B = \frac{|e|\hbar}{2m}$

We choose the z-axis along the magnetic field $\vec{B}$. Then for the orbital quantum number m and the spin quantum number m_s ($L_z = \hbar m$, $S_z = \hbar m_s$), we have the energy

$$E(m, m_s) = \mu_B B\,(m + g\,m_s)$$

For $l=1$ we have $m = -1, 0, 1$; on the other hand, $m_s = \pm\frac{1}{2}$.

Then, if we set $g = 2$, we obtain the energy levels

$2\mu_B B$ ——— $(m = 1,\ m_s = \frac{1}{2})$

$\mu_B B$ ——— $(m = 0,\ m_s = \frac{1}{2})$

0 ——— $(m = 1,\ m_s = -\frac{1}{2})$ <u>and</u> $(m = -1,\ m_s = \frac{1}{2})$

$-\mu_B B$ ——— $(m = 0,\ m_s = -\frac{1}{2})$

$-2\mu_B B$ ——— $(m = -1,\ m_s = -\frac{1}{2})$

Total 6 states, but 5 energy levels; one of them is degenerate: two states have the same energy

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3. (a) Quarks have spin $s = 1/2$. The total spin is $\vec{S} = \vec{S}_1 + \vec{S}_2$. If the spins are parallel, $s = s_1 + s_2 = 1$. If they are antiparallel, $s = 0$. The values of s generally vary from $s_1 + s_2$ to $|s_1 - s_2|$. In our case we have $m_s = 1, 0, -1$ for $s = 1$ and $m_s = 0$ for $s = 0$. There are 4 states.

On the other hand, each of the spins of individual quarks takes on 2 values, and therefore the total number of states for two quarks is $2 \times 2 = 4$.

(b) In class we found the wave function

$$\Psi(t) = a_+ \chi_+ e^{-iE_+ t/\hbar} + a_- \chi_- e^{-iE_- t/\hbar}, \quad E_\pm = \pm \mu_B B$$

$$\langle S_y(t) \rangle = \langle \Psi(t) | S_y | \Psi(t) \rangle; \text{ we have } S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and therefore $\langle \chi_+ | S_y | \chi_+ \rangle = \langle \chi_- | S_y | \chi_- \rangle = 0$, $\langle \chi_+ | S_y | \chi_- \rangle = -i\hbar/2$, and $\langle \chi_- | S_y | \chi_+ \rangle = i\hbar/2$, so that

$$\langle \Psi(t) | S_y | \Psi(t) \rangle = \frac{\hbar}{2}\left[-i a_+^* a_- e^{i(E_+ - E_-)t/\hbar} + i a_+ a_-^* e^{-i(E_+ - E_-)t/\hbar} \right]$$

$$= |a_+ a_-| \hbar \sin(\omega_L t + \varphi) \quad \text{with } \omega_L = \frac{|e| B}{m} = \frac{E_+ - E_-}{\hbar}$$

and $\varphi = \arg a_+^* a_-$

$\Rightarrow$ complete agreement with $\langle E_x(t) \rangle$ found in class and the solution of problem 1

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4. The expression for the energy splitting is

$$\Delta E = \mu_B g B \simeq 2\mu_B B = \frac{|e|\hbar}{m} B$$

cf. problem 3; the splitting comes from the electron spin.

We have $|e| = 1.6 \cdot 10^{-19}$ C, $m = 0.91 \cdot 10^{-30}$ kg, $\hbar = 6.63 \cdot 10^{-34}/2\pi$

For $B = 1$ T we then have

$$\Delta E \simeq 1.85 \cdot 10^{-23}\ J \simeq 1.12 \cdot 10^{-4}\ eV \simeq 2.8 \cdot 10^{10}\ Hz$$