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Problem Set 5

1. Use that $[r_{1\alpha}, P_{1\beta}] = i\hbar\,\delta_{\alpha\beta}$, $[r_{1\alpha}, P_{2\beta}] = 0$, $[P_{1\alpha}, r_{2\beta}] = 0$ $\Rightarrow$ coordinates and momenta of different particles commute.

Then

$$[r_\alpha, P_\beta] \equiv \left[r_{1\alpha} - r_{2\alpha}, \frac{m_2 P_{1\beta} - m_1 P_{2\beta}}{M}\right] = \frac{m_2}{M}[r_{1\alpha}, P_{1\beta}] + \frac{m_1}{M}[r_{2\alpha}, P_{2\beta}]$$

$$= \frac{m_1 + m_2}{M}\, i\hbar\,\delta_{\alpha\beta} = i\hbar\,\delta_{\alpha\beta}$$

$$[R_\alpha, P_\beta] = \left[\frac{m_1 r_{1\alpha} + m_2 r_{2\alpha}}{M}, P_{1\beta} + P_{2\beta}\right] = \frac{m_1}{M}[r_{1\alpha}, P_{1\beta}] + \frac{m_2}{M}[r_{2\alpha}, P_{2\beta}]$$

$$= \frac{m_1 + m_2}{M}\, i\hbar\,\delta_{\alpha\beta} = i\hbar\,\delta_{\alpha\beta}$$

$$[r_\alpha, P_\beta] = [r_{1\alpha} - r_{2\alpha}, P_{1\beta} + P_{2\beta}] = [r_{1\alpha}, P_{1\beta}] - [r_{2\alpha}, P_{2\beta}] = 0$$

$$[P_\alpha, R_\beta] = \left[\frac{m_2 P_{1\alpha} - m_1 P_{2\alpha}}{M}, \frac{m_1 r_{1\beta} + m_2 r_{2\beta}}{M}\right]$$

$$= \frac{m_1 m_2}{M^2}[P_{1\alpha}, r_{1\beta}] - \frac{m_1 m_2}{M^2}[P_{2\alpha}, r_{2\beta}] =$$

$$= \frac{m_1 m_2}{M^2}\left[-i\hbar\,\delta_{\alpha\beta} + i\hbar\,\delta_{\alpha\beta}\right] = 0$$

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2. We have to express $\vec{p}_1, \vec{p}_2, \vec{r}_1, \vec{r}_2$ in terms of $\vec{r}, \vec{p}, \vec{R}, \vec{P}$.

For example, we have equations

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M} \qquad (*)$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

| first, multiply by $\frac{m_2}{M}$ and add to $(*)$
next multiply by $\frac{m_1}{M}$ and subtract from $(*)$

$$\Rightarrow \vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}, \qquad \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$$

Similarly for $\vec{p}$ and $\vec{P} \Rightarrow$ $\vec{p}_1 = \vec{p} + \frac{m_1}{M}\vec{P}$, $\vec{p}_2 = -\vec{p} + \frac{m_2}{M}\vec{P}$

Then the Hamiltonian is

$$H = \frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2^{\,2}}{2m_2} + U(|\vec{r}_1 - \vec{r}_2|) = \frac{1}{2m_1}\left(\vec{p} + \frac{m_1}{M}\vec{P}\right)^2 + \frac{1}{2m_2}\left(\frac{m_2}{M}\vec{P} - \vec{p}\right)^2 + U(r)$$

$$= \frac{1}{2}\vec{p}^{\,2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) + \frac{1}{2}\vec{P}^2\left(\frac{m_1^2}{m_1 M^2} + \frac{m_2^2}{m_2 M^2}\right) + \left(\frac{m_1}{m_1 M}\vec{p}\cdot\vec{P} - \frac{m_2}{m_2 M}\vec{p}\cdot\vec{P}\right) + U(r)$$

$$= \frac{1}{2\mu}\vec{p}^{\,2} + \frac{1}{2M}\vec{P}^2 + U(r), \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

We have used here that $[p_{1\alpha}, p_{2\beta}] = 0$ and $[p_\alpha, P_\beta] = 0$. We also used that

$$p_{1x} = -i\hbar\frac{\partial}{\partial x_1} = -i\hbar\frac{\partial}{\partial x}\frac{\partial x}{\partial x_1} - i\hbar\frac{\partial}{\partial X}\frac{\partial X}{\partial x_1} = -i\hbar\frac{\partial}{\partial x} - i\hbar\frac{m_1}{M}\frac{\partial}{\partial X}$$

$$= p_x + \frac{m_1}{M}P_x,$$ and similarly for other components of the momentum

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3. For the hydrogen atom we can think that, in the previous problem, particle 2 is an electron, $m_2 = m$, whereas particle 1 is a proton, $m_1 = m_p$. Then $M = m + m_p$, $\mu = \frac{m \cdot m_p}{m + m_p}$.

In the variables $(\vec{r}, \vec{p})$, $(\vec{R}, \vec{P})$, we can think of two particles, one with Hamiltonian $H_M = \frac{\vec{P}^2}{2M}$, coordinate $\vec{R}$, and momentum $\vec{P}$. The other has coordinate $\vec{r}$, momentum $\vec{p}$, and mass μ; its Hamiltonian is

$$H_\mu = \frac{\vec{p}^{\,2}}{2\mu} + U(r) = \frac{\vec{p}^{\,2}}{2\mu} - \frac{e^2}{4\pi\varepsilon_0 r}.$$

The total Hamiltonian is $H = H_M + H_\mu$. An eigenfunction is

$$\Psi(\vec{r}, \vec{R}) = C \exp\left(i \frac{\vec{P}\cdot\vec{R}}{\hbar}\right) \psi_\mu(\vec{r}), \qquad H_\mu \psi_n(\vec{r}) = E_n \psi_n(\vec{r}).$$

$$H\Psi(\vec{r}, \vec{R}) = \left(\frac{P^2}{2M} + E_n\right) \Psi(\vec{r}, \vec{R}).$$

One of the auxiliary particles (with coordinate $\vec{R}$) is just the hydrogen atom. It moves with momentum $\vec{P}$. The Hamiltonian H_μ is of the same form as the Hamiltonian of the electron in the hydrogen atom if the proton is assumed to be at $r=0$ and does not move. The only difference is that the electron mass is replaced with μ. In the ground state

$$\psi_1(r) = C' \exp(-r/\tilde{a}), \qquad \tilde{a} = \frac{4\pi\varepsilon_0 \hbar^2}{\mu e^2}$$

The energy of the ground state of the atom corresponds to $P = 0$, $E = E_1 = -\frac{\hbar^2}{2\mu \tilde{a}^2}$

Note that $\frac{m - \mu}{m} \simeq \frac{m}{m_p} \sim \frac{1}{2000}$ - a small difference

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4. As in the previous problem, except that now $M = 2m$ and $\mu = \frac{m^2}{2m} = \frac{m}{2}$.

The ground state energy is $E_1 = -\frac{\hbar^2}{2\mu\tilde{a}^2} = \frac{1}{2}(E_1)_{\text{hydrogen atom}}$