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## Problem Set 6

1.(a) If $x_i^{(0)}$ is a nondegenerate (isolated) root of $F_0(x)$, the expansion of $F_0(x)$, $F_1(x)$ in $\delta x = x_i - x_i^{(0)}$ has the form

$F_0(x) \simeq F_0' \,\delta x + \frac{1}{2} F_0'' (\delta x)^2$; $\quad F_1(x) \simeq F_1(x_i^{(0)}) + F_1' \,\delta x + \ldots$

where prime indicates a derivative and all derivatives are evaluated at $x_i^{(0)}$.

We seek the solution of equation $F_0(x) + \lambda F_1(x) = 0$ as a series $\delta x = \lambda\, \delta x^{(1)} + \lambda^2 \delta x^{(2)}$

First order in $\lambda$: $\quad F_0' \,\delta x^{(1)} + F_1 = 0$, $\quad \boxed{\delta x^{(1)} = -\frac{F_1}{F_0'}}$;

Second order:

$$F_0' \,\delta x^{(2)} + \frac{1}{2} F_0'' (\delta x^{(1)})^2 + F_1' \,\delta x^{(1)} = 0$$

$$\boxed{\delta x^{(2)} = -\frac{1}{F_0'}\left[\frac{1}{2}\frac{F_0'' F_1^2}{(F_0')^2} - \frac{F_1' F_1}{F_0'}\right]}$$

(b) Write $18 = 4^2 + \lambda$, then set $\lambda = 0$; $\sqrt{18} = 4 + \delta x$

$16 + 8\,\delta x + \delta x^2 = 18 = 16 + \lambda$; $\quad \delta x = \lambda \delta x^{(1)} + \lambda^2 \delta x^{(2)} + \lambda^3 \delta x^{(3)}$

$8\,\delta x^{(1)} = 1$; $\quad 8\,\delta x^{(2)} + \delta x^{(1)2} = 0$; $\quad 8\,\delta x^{(3)} + 2\,\delta x^{(1)} \delta x^{(2)} = 0$

$\delta x^{(1)} = \frac{1}{8}$; $\quad \delta x^{(2)} = \frac{-1}{512}$; $\quad \delta x^{(3)} = \frac{1}{8}\,\frac{1}{2048}$

$\delta x \simeq \frac{1}{4} - \frac{1}{128} + \frac{1}{2048} \simeq 0.2427$; $\quad \sqrt{18} \simeq 4.2427$

Write $\log_2 5 = \frac{\ln 5}{\ln 2}$; $\quad \ln 5 = \ln(4+1) = \ln\left[4\left(1+\frac{1}{4}\right)\right] = \ln 4 + \ln\left(1+\frac{1}{4}\right)$

$\ln\left(1+\frac{1}{4}\right) \simeq \frac{1}{4} - \frac{1}{2}\,\frac{1}{16} + \frac{1}{3}\,\frac{1}{64} \simeq 0.224$

$\ln 2 \simeq \ln[e - 1.72] = \ln\left[e\left(1 - \frac{1.72}{e}\right)\right] = 1 - \frac{1.72}{e} - \frac{1}{2}\left(\frac{1.72}{e}\right)^2 - \frac{1}{3}\left(\frac{1.72}{e}\right)^3$

$- \frac{1}{4}\left(\frac{1.72}{e}\right)^4 \simeq 0.693$;

$\log_2 5 \simeq 2 + \frac{0.224}{0.693} \simeq 2.32$

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2. The energy of the ground state of a harmonic oscillator is $\hbar\omega_0/2$, if the energy is counted off from the bottom of the potential well. In the present case $\min U(x)=0$, so, the energy is $\hbar\omega_0/2$, with $\omega_0 = \sqrt{\kappa/M}$.

Bosons. In the lowest-energy state all of them are in the ground state of the harmonic oscillator. $E_\ell^{(Bosons)} = 3\hbar\omega_0/2$.

In the excited state two bosons are in the ground oscillator state and one is in the first excited oscillator state (the energy of the first excited oscillator state is $\frac{3}{2}\hbar\omega_0$). Then

$E_e^{(Bosons)} = \frac{5}{2}\hbar\omega_0$

Fermions. In the lowest-energy state we have two fermions with opposite spins in the ground oscillator state and one fermion in the first excited oscillator state. The energy is

$$E_\ell^{(fermions)} = 2\cdot\frac{1}{2}\hbar\omega_0 + \frac{3}{2}\hbar\omega_0 = \frac{5}{2}\hbar\omega_0$$

In the first excited state, there are two options: 2 fermions in the ground oscillator state, 1 in the second excited state, or 1 fermion in the ground state and 2 in the first excited oscillator state. The states have the same energy

$$\left(E_e^{(fermions)}\right)_1 = 2\cdot\frac{1}{2}\hbar\omega_0 + \frac{5}{2}\hbar\omega_0 = \frac{7}{2}\hbar\omega_0$$

$$\left(E_e^{fermions}\right)_2 = \frac{1}{2}\hbar\omega_0 + 2\cdot\frac{3}{2}\hbar\omega_0 = \frac{7}{2}\hbar\omega_0$$

Such states are called degenerate (you do not know this yet).

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3. The wave function should not change if we interchange any two particles. Such function is given by the sum of all permutations

$$\Psi(\vec{r}_1, m_{s1}; \vec{r}_2, m_{s2}; \vec{r}_3, m_{s3}) = C \sum_{P_1, P_2, P_3} \psi_{P_1}(\vec{r}_1, m_{s1}) \psi_{P_2}(\vec{r}_2, m_{s2}) \psi_{P_3}(\vec{r}_3, m_{s3})$$

where $P_i$ runs over the states $a$, $b$, and $c$, and all $P_i$ are different. For example, $P_1 = a$, $P_2 = b$, $P_3 = c$, or $P_1 = a$, $P_2 = c$, $P_3 = b$, or $P_1 = b$, $P_2 = a$, $P_3 = c$, etc.

There are 6 terms in the sum, and therefore

$$C = \frac{1}{\sqrt{6}}$$

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4. It is convenient to use the raising and lowering operators $a^+$ and $a$.

$$x = \sqrt{\frac{\hbar}{2M\omega}}\,(a+a^+), \quad p = -i\sqrt{\frac{\hbar M\omega}{2}}\,(a-a^+), \quad \omega = \sqrt{\frac{k}{M}}$$

The zeroth order Hamiltonian is $H^{(0)} = \frac{p^2}{2M} + \frac{1}{2}kx^2$

or $H^{(0)} = \hbar\omega\left(a^+a + \frac{1}{2}\right)$.

The zeroth-order eigenfunctions are the eigenfunctions of the harmonic oscillator, $\psi_n^{(0)} \equiv |n\rangle$. We have

$$\langle n|a|m\rangle = \sqrt{m}\,\delta_{n,m-1}, \quad \langle n|a^+|m\rangle = \sqrt{m+1}\,\delta_{n,m+1}$$

$$E_n^{(0)} = \hbar\omega\left(n+\frac{1}{2}\right)$$

The perturbation is $H^{(1)} = \frac{1}{4}\gamma x^4 = \gamma\frac{\hbar^2}{16M^2\omega^2}(a+a^+)^4$

The first-order correction to the energy is

$E_n^{(1)} = \langle n|H^{(1)}|n\rangle$. We need to keep in $H^{(1)}$ only the terms with the equal number of the operators $a$ and $a^+$, the diagonal matrix element of other terms is zero.

Then we have

$$E_n^{(1)} = \frac{\gamma\hbar^2}{16M^2\omega^2}\langle n|(a^2+a^{+2}+2a^+a+1)(a^2+a^{+2}+2a^+a+1)|n\rangle$$

$$= \frac{\gamma\hbar^2}{16M^2\omega^2}\langle n|a^2a^{+2}+a^{+2}a^2+(2a^+a+1)^2|n\rangle$$

$$= \frac{\gamma\hbar^2}{16M^2\omega^2}\left[(n+1)(n+2)+n(n-1)+(2n+1)^2\right] =$$

$$= \frac{3\gamma\hbar^2}{16M^2\omega^2}(2n^2+2n+1)$$