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Problem Set 7

1. The wave function should change sign when any two fermions are interchanged. A sign change is a feature of a determinant. If we write the wave function as

$$\Psi_{abc}(\bar{r}_1, m_{s1};\ \bar{r}_2, m_{s2};\ \bar{r}_3, m_{s3}) = C \begin{vmatrix} \psi_a(\bar{r}_1, m_{s1}) & \psi_a(\bar{r}_2, m_{s2}) & \psi_a(\bar{r}_3, m_{s3}) \\ \psi_b(\bar{r}_1, m_{s1}) & \psi_b(\bar{r}_2, m_{s2}) & \psi_b(\bar{r}_3, m_{s3}) \\ \psi_c(\bar{r}_1, m_{s1}) & \psi_c(\bar{r}_2, m_{s2}) & \psi_c(\bar{r}_3, m_{s3}) \end{vmatrix}$$

then the interchages $(\bar{r}_3, m_{s3}) \leftrightarrows (\bar{r}_2, m_{s2})$, $(\bar{r}_1, m_{s1}) \rightleftarrows (\bar{r}_2, m_{s2})$, $(\bar{r}_1, m_{s1}) \rightleftarrows (\bar{r}_3, m_{s3})$ lead to the change of sign.

There are 6 terms in the determinant; for example,

$-\psi_a(\bar{r}_1, m_{s1})\,\psi_b(\bar{r}_3, m_{s3})\,\psi_c(\bar{r}_2, m_{s2})$.

All these terms are orthogonal and normalized to 1, like $\sum_{m_{s1}, m_{s2}, m_{s3}} \int d^3\bar{r}_1\, d^3\bar{r}_2\, d^3\bar{r}_3 \left|\psi_a(\bar{r}_1, m_{s1})\,\psi_b(\bar{r}_3, m_{s3})\,\psi_c(\bar{r}_2, m_{s2})\right|^2$

$= 1$.

Therefore $C = \frac{1}{\sqrt{6}} = \frac{1}{\sqrt{3!}}$.

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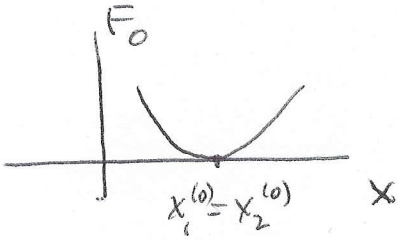
2. For a degenerate root, $x_1^{(0)} = x_2^{(0)}$, we have $F_0'(x_1^{(0)}) = 0$

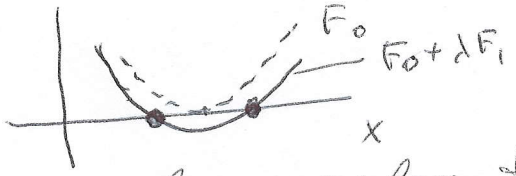


Writing the expression for the root as $x_i = x_1^{(0)} + \delta x$ and using $F_0(x) \simeq \frac{1}{2} F_0''(\delta x)^2$,

from equation $F_0(x) + \lambda F_1(x) = 0$ we have

$$\frac{1}{2} F_0'' \delta x^2 + \lambda F_1(x_1^{(0)}) = 0, \qquad \delta x = \pm \sqrt{\frac{-2\lambda F_1(x_1^{(0)})}{F_0''(x_1^{(0)})}}$$

$\Rightarrow$ the root is split, as shown in the figure.



Note that real roots exist only for $\lambda F_1(x_1^{(0)}) F_0''(x_1^{(0)}) < 0$.

The analysis applies to complex roots as well.

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3. For a single fermion, the orbital part of the ground-state wave function is

$$\psi_0(x) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega_0 x^2}{2\hbar}\right], \quad \omega_0 = \sqrt{\frac{k}{m}}, \quad E_0 = \frac{\hbar\omega_0}{2}$$

Because the fermions can have different spin components, their wave function in the absence of the perturbation is (in the ground state and besides the spin-dependent part)

$$\psi_g(x_1, x_2) = \psi_0(x_1)\psi_0(x_2), \quad E_g^{(0)} = 2E_0 = \hbar\omega_0$$

The correction to the energy due to the perturbation is

$$E^{(1)} = \int dx_1 dx_2 \, \psi_g^*(x_1, x_2) V(x_1, x_2) \psi_g(x_1, x_2)$$

$$= -a \int dx_1 \psi_0^4(x_1) = -a \frac{m\omega_0}{\pi\hbar} \int dx_1 e^{-2m\omega_0 x^2/\hbar} = -a\sqrt{\frac{m\omega_0}{2\pi\hbar}}$$

Dimension: $[a] = J \cdot m$, $[E^{(1)}] = J$ - correct

For the perturbation theory to apply, $|E^{(1)}|$ should be small compared to the distance between the energy levels - this is clear, because $E_n^{(2)} \sim \frac{(E^{(1)})^2}{\underbrace{E_n^{(0)} - E_m^{(0)}}_{\text{inter level distance}}}$, and we need $|E^{(2)}| \ll |E^{(1)}|$. Then $|E^{(1)}| \ll \hbar\omega_0$, $|a| \ll \hbar^{3/2}(\omega_0/m)^{1/2}$

Equivalently, can say that $|E^{(1)}|$ is a small correction compared to the inter level distance $\sim \hbar\omega_0$.

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4. The single-particle states are $\Psi(x, m_s) = \psi_n(x)\,|m_s\rangle$ where $\psi_n(x)$ is the wave function of the harmonic oscillator on the nth level. The energy is $E_{n,m_s} = \hbar\omega_0(n+\frac{1}{2}) - \gamma\hbar m_s B$.

If $\hbar\omega_0 > \gamma\hbar B$, the ground state of the fermions in the absence of interaction is

$$\Psi^{(0)} = \psi_0(x_1)\psi_0(x_2)\cdot\frac{1}{\sqrt{2}}\left(\left|m_{s_1}=\frac{3}{2}, m_{s_2}=\frac{1}{2}\right\rangle - \left|m_{s_1}=\frac{1}{2}, m_{s_2}=\frac{3}{2}\right\rangle\right)$$

The correction due to the interaction is the same as in the previous problem, i.e.

$$E_g^{(0)} = \hbar\omega_0 - 2\gamma\hbar B, \qquad E^{(1)} = -a\sqrt{\frac{m\omega_0}{2\pi\hbar}}$$

If $\hbar\omega_0 < \gamma\hbar B$, the ground state in the absence of interaction is

$$\Psi^{(0)} = \left|m_{s_1}=\frac{3}{2}, m_{s_2}=\frac{3}{2}\right\rangle \frac{1}{\sqrt{2}}\left[\psi_0(x_1)\psi_1(x_2) - \psi_0(x_2)\psi_1(x_1)\right]$$

$$E_g^{(0)} = -3\gamma\hbar B + 2\hbar\omega_0$$

$$E^{(1)} = \frac{1}{2}(-a)\int dx_1 dx_2 \left[\psi_0(x_1)\psi_1(x_2) - \psi_0(x_2)\psi_1(x_1)\right]^2 \delta(x_1 - x_2) = 0.$$

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5. One can think of a system as being confined to one dimension, if the energy to excite motion in other dimensions are much higher than the energies of interest. Imagine that, in the previous problems, the full potential has the form

$$U(x,y,z) = \frac{Kx^2}{2} + \frac{1}{2}K(y^2+z^2), \text{ with } K \gg k.$$

Exciting vibrations along y or z axes would require energy $\hbar(K/m)^{1/2}$. If this energy is larger than the energies of interest, including $\hbar(k/m)^{1/2}$ and $\gamma\hbar B$, the motion can be thought of as one-dimensional. Ultimately, for very high energies, the motion is three-dimensional.