PHY 472-2020
Problem Set 11.

1. We can either solve the time-dependent Schrödinger equation directly or use the result of the degenerate perturbation theory

$$\Psi(t) = C_+\left[\alpha_{1+}|\psi_1\rangle + \alpha_{2+}|\psi_2\rangle\right]e^{-iE_+t/\hbar}$$
$$+ C_-\left[\alpha_{1-}|\psi_1\rangle + \alpha_{2-}|\psi_2\rangle\right]e^{-iE_-t/\hbar} \quad \text{for } t>0$$

For $E_1^{(0)} = E_2^{(0)} \equiv E^{(0)}$ we have $E_\pm = \pm|V_{12}| + E^{(0)}$. Use the coefficients $\alpha_{1\pm}, \alpha_{2\pm}$ found earlier, $\alpha_{1+} = \frac{1}{\sqrt{2}}$, $\alpha_{2+} = \frac{1}{\sqrt{2}}\frac{V_{12}^*}{|V_{12}|}$; $\alpha_{1-} = \frac{1}{\sqrt{2}}$, $\alpha_{2-} = -\frac{1}{\sqrt{2}}\frac{V_{12}^*}{|V_{12}|}$

The coefficients $C_\pm$ should be found from the normalization condition and the initial condition:

$$C_+\alpha_{1+} + C_-\alpha_{1-} = 1, \quad C_+\alpha_{2+} + C_-\alpha_{2-} = 0 \quad \text{(initial condition)}$$

$\Rightarrow C_- = C_+ = \frac{1}{\sqrt{2}}$. Satisfies the normalization condition

We used the orthogonality of the wave functions
$\alpha_{1+}|\psi_1\rangle + \alpha_{2+}|\psi_2\rangle$ and $\alpha_{1-}|\psi_1\rangle + \alpha_{2-}|\psi_2\rangle$

Alternatively, set $\Psi(t) = \left[A_1(t)|\psi_1\rangle + A_2(t)|\psi_2\rangle\right]e^{-iE^{(0)}t/\hbar}$

For $t>0$:

$$i\hbar\dot{A}_1 = V_{12}A_2, \quad i\hbar\dot{A}_2 = V_{21}A_1;$$

$$i\hbar\ddot{A}_1 = \frac{1}{i\hbar}|V_{12}|^2A_1 \Rightarrow A_1(t) = B_{1+}e^{-i|V_{12}|t/\hbar} + B_{1-}e^{i|V_{12}|t/\hbar}$$

$$A_2(t) = \frac{i\hbar}{V_{12}}\dot{A}_1 = \frac{|V_{12}|}{V_{12}}B_{1+}e^{-i|V_{12}|t/\hbar} - \frac{|V_{12}|}{V_{12}}B_{1-}e^{i|V_{12}|t/\hbar}$$

$A_2(0) = 0 \Rightarrow B_{1-} = B_{1+}$; Normalization: $|A_1|^2 + |A_2|^2 = 1$

$\Rightarrow 2\left(|B_{1+}|^2 + |B_{1-}|^2\right) = 1$, $B_{1\pm} = \frac{1}{2}$

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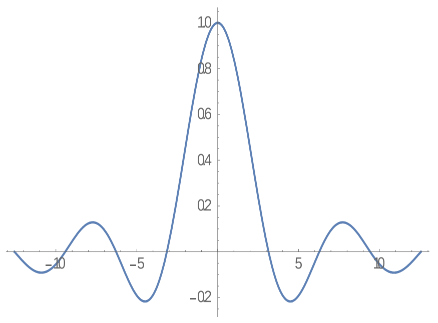
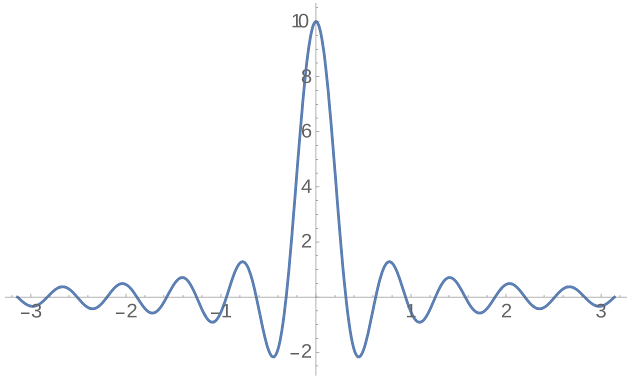
2. The fist and the second expressions coincide. Show only the second expression;

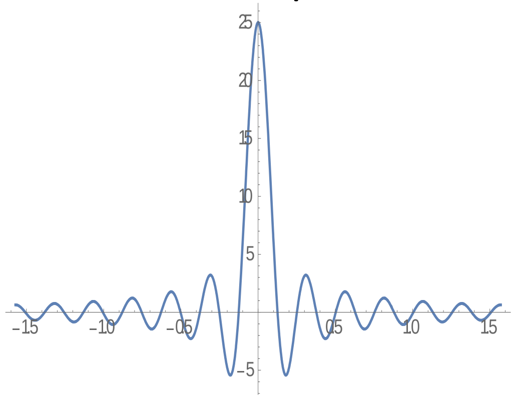
sin x/x



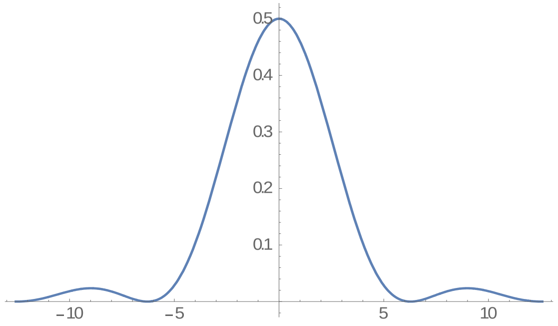
sin 10 x/x



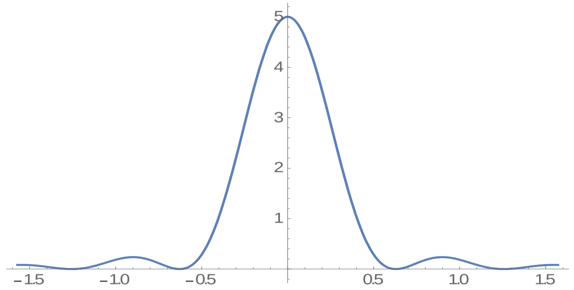
sin 25x/x



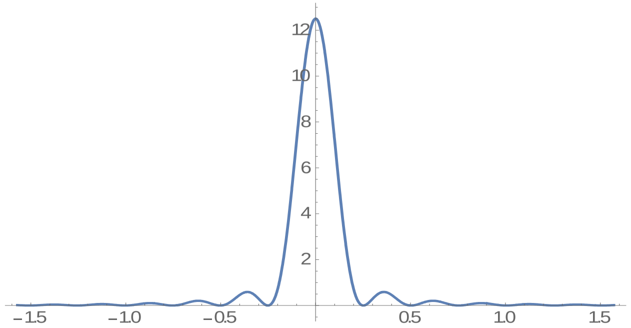
$(1-\cos x)/x^2$



$(1-\cos 10\,x)\;/\;10\,x^2$



$(1-\cos 25\,x)\;/\;25\,x^2$



To prove: have to show that the functions are zero for $x \neq 0$ and that the integral of the functions are 1 for function 1 and π for functions 2 and 3. For function 1 you know the result from the theory of the Fourier transform. Function 2 is the same as function 1 (except for the factor π). For function 3 the easiest way is to write the condition as $[1-\cos xT]/x^2 \to \pi T\ \delta(x)$ for $T \to \infty$. Differentiate over T and use function 2 to see the result.

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3. The wave function of the bound state

$$\psi_0(x) = \sqrt{\varkappa}\, e^{-\varkappa|x|}, \quad \varkappa = \frac{m\alpha}{\hbar^2}, \quad E_0 = -\frac{m\alpha^2}{2\hbar^2}$$

Wave functions of the quasicontinuous spectrum are

$\psi_k(x) = \frac{1}{\sqrt{L}} e^{ikx}$. The transition rate is

$$W_0 = \frac{\pi F^2}{2\hbar} \sum_k |\langle\psi_0|x|\psi_k\rangle|^2 \delta(E_k - E_0 - \hbar\omega)$$

$$= \frac{\pi}{2\hbar} F^2 \frac{L}{2\pi} \int dk\, \delta(E_k - E_0 - \hbar\omega) \frac{\varkappa}{L} \left| \int_{-\infty}^{\infty} dx\, e^{-\varkappa|x| + ikx} x \right|^2$$

the transition $\sum_k \to \int dk$ was discussed in class

normalization

more carefully: $\int_{-L/2}^{L/2}$, but L is large

$$\int_{-\infty}^{\infty} dx\, e^{-\varkappa|x|+ikx} x = 2i\,\mathrm{Im} \int_0^{\infty} dx\, e^{-\varkappa x + ikx} x = 2i\,\mathrm{Im}\frac{1}{(\varkappa - ik)^2} = \frac{4ik\varkappa}{(k^2+\varkappa^2)^2}$$

We have $E_k = \frac{\hbar^2 k^2}{2m}$; $\int_{-\infty}^{\infty} dk\, \delta(E_k - E_0 - \hbar\omega) f(k) \Big|_{f(k)=f(-k)} =$

$$= 2\int_0^{\infty} dk\, f(k)\, \delta(E_k - E_0 - \hbar\omega) = \frac{2f(k_\omega)}{(\partial E_k/\partial k)_{k_\omega}},$$ where k_ω is given by

the condition $\frac{\hbar^2 k_\omega^2}{2m} = E_0 + \hbar\omega = \hbar\omega - \frac{m\alpha^2}{2\hbar^2}$, $k_\omega > 0$

$\frac{\partial E_k}{\partial k} = \frac{\hbar^2 k}{m}$

We have $f(k) = \left| \int_{-\infty}^{\infty} dx\, e^{-\varkappa|x|+ikx} x \right|^2 = \frac{16k^2\varkappa^2}{(k^2+\varkappa^2)^4}$,

Then the transition rate is

$$W_0 = \frac{8k_\omega \varkappa^3 m F^2}{\hbar^3 (k_\omega^2 + \varkappa^2)^4}.$$

Check dimension! Should be s^{-1}.

$$k_\omega = \sqrt{\frac{2m}{\hbar^2}\left(\hbar\omega - \frac{m\alpha^2}{2\hbar^2}\right)}$$

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4. Change variables, $\tilde{\vec{r}} = \vec{r} - \vec{r}\,'$; $\partial_{\vec{r}} = \partial_{\tilde{\vec{r}}}$. To simplify notations I will skip the tilde, that is, set $\vec{r}\,' = \vec{0}$. Then we have to prove that $\nabla^2 \frac{1}{r} = -4\pi\delta(\vec{r})$.

In spherical coordinates

$$\nabla^2 \frac{1}{r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\frac{1}{r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\frac{1}{r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\frac{1}{r}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(-\frac{1}{r^2}\right)\right) = 0 \quad \text{for } r > 0.$$

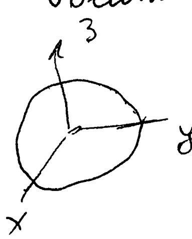
$r = 0$ is a singular point, $\frac{1}{r} \to \infty$ for $r \to 0$. We have to consider it differently.

Use $\nabla^2 \frac{1}{r} = \vec{\nabla}\cdot\left(\vec{\nabla}\frac{1}{r}\right)$ and the Gauss theorem:

$$\int \vec{\nabla}\cdot\vec{G}\, dV = \oint \vec{G}\cdot d\vec{A}$$

integral of a divergence of vector $\vec{G}$ over a volume V

integral over the surface that encloses the volume V



Take a sphere of radius r around the origin. $d\vec{A} = r^2 d\Omega\, \hat{r}$, Ω is the solid angle, $\hat{r}$ is a unit vector in the radial direction. In our case $\vec{G} = \vec{\nabla}\frac{1}{r}$. The radial component of the gradient is $\hat{r}\frac{\partial}{\partial r}$,

$$G_r = \frac{\partial}{\partial r}\frac{1}{r} = -\frac{1}{r^2}. \qquad \oint \vec{G}\cdot d\vec{A} = -\int \frac{1}{r^2} r^2 d\Omega = -4\pi.$$

Therefore $\nabla^2 \frac{1}{r} = -4\pi\delta(\vec{r})$.